Stability Analysis of Dynamical Neural Networks

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***Abstract*—In this paper, we use the matrix measure technique to study stability of dynamical neural networks. Testable conditions for global exponential stability of nonlinear dynamical systems and dynamical neural networks are given. It shows how a few well-known results can be unified and generalized in a straightforward way. Local exponential stability of a class of dynamical neural networks is also studied; we point out that the local exponential stability of any equilibrium point of dynamical neural networks is equivalent to the stability of the linearized system around that equilibrium point. From this, some well-known and new, sufficient conditions for local exponential stability of neural networks are obtained.**

I. Introduction

HOPFIELD-TYPE (additive) neural networks have been intensively studied in the past decade and have been applied to optimization problems and specific problems of A/D converter design ([5], [6], [10], [12], and the references therein). Such engineering applications rely crucially on the analysis of the neurodynamics and the behavior of dynamical neural networks. Therefore, the analysis of the neurodynamics and behavior such as stability and oscillation is indispensable for practical design of neural networks.

The qualitative properties of dynamical neural networks (notably Hopfield-type neural networks) have been intensively investigated by Michel and his colleagues [4], [10]–[12]. They applied large-scale system techniques to obtain a large set of sufficient conditions for local asymptotic (exponential) stability for a few classes of dynamical neural networks. However, a problem normally encountered in this approach is the existence of more than one equilibrium point which may correspond to local minima. Global properties are hard to extract, and a global minimum is difficult to achieve. To overcome this dilemma, it is desirable to design neural networks which have only one unique equilibrium point and are globally stable (or attractive) so that the global property can be extracted. In this case, we do not need to specify the initial conditions of neural circuits, since all trajectories starting from anywhere will settle down to the same unique equilibrium. This equilibrium depends only on the external stimuli. In fact, if the interconnection among individual neurons, the time constants of the circuits, and the activation functions are fixed, we obtain a mapping from the external stimuli space to the activation space. Moreover, unlike the Winner-Take-All circuits where resetting of activations has to be made whenever input stimuli change [1], if neural circuits are globally stable for each external stimulating input, we need not reset the activations when changing inputs. This is convenient for a neural circuit running in real time. Hirsch [1] noted the importance of global stability or global attractiveness and obtained a few sufficient conditions using Gershgorin's circle theorem. Kelly [2] applied the contraction mapping technique to obtain some sufficient conditions for global stability. Matsuoka [13] generalized some of Hirsch's and Kelly's results using a new Lyapunov function. Recently, Kaszkurewicz and Bhaya [3] proved that the diagonal stability of the interconnection matrix implies the existence and uniqueness of an equilibrium and global stability of the equilibrium. Forti *et al.* [9] showed that the negative semidefiniteness of the interconnection matrix guarantees the global stability of a Hopfield network with a certain robustness property. Yang and Dillon [7] also obtained some sufficient conditions for the local exponential stability and for the existence and uniqueness of an equilibrium point. However, they failed to give any conditions for global stability, because there exists such a system that although it has a unique equilibrium point and it is locally (exponentially) stable, it may not be globally stable.

This paper addresses both the global stability and local stability of dynamical neural networks. The matrix measure technique has been used to study the stability property of nonlinear dynamical systems and many well-known results have been unified and generalized. Some general sufficient conditions for nonlinear dynamical systems have been obtained and have been applied to a class of asymmetrical Hopfield-type networks.

The paper is organized as follows. In the second part, we present some preliminaries on matrix measure. Properties of matrix measure, old or new, are summarized. Coppel's inequality and its relationship to the estimation and stability of linear systems are discussed. In the third section, we generalize the stability results of linear systems in terms of matrix measure to nonlinear dynamical systems. Applying these new results to a class of dynamical neural networks, we obtain a set of conditions for global as well as local exponential stability of the neural networks. We also illustrate the unification of some well-known results in this area. We conclude this paper in the last section and point out future research directions in this field.

II. Properties of Matrix Measure

Matrix measure has been used to study the error bounds in the numerical integration of ordinary differential equations [19], [20], [22] and estimation and stability of solutions of differential equations [21]. Recently, Fang *et al.* [18]

Manuscript received October 29, 1994; revised April 20, 1995. This work was supported in part by the College of Engineering at Boston University.

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Publisher Item Identifier S 1045-9227(96)04400-1.

revitalized this technique to study the robust stability of interval linear dynamical systems. In this paper, we will extend this technique to the study of dynamical neural networks for global or local pattern formation.

Let $|x|$ denote a vector norm of x on C^n, and $||A||$ is the matrix norm of A induced by the vector norm $|\cdot|$. $\mu(A)$ is the matrix measure of A defined as

$$\mu(A) \triangleq \lim_{\theta \downarrow 0^+} \frac{||I+\theta A|| - 1}{\theta}$$

where I is the identity matrix with the same dimension as A. For convenience, we collect only the important properties of matrix measure, some of which are new in the following lemma. More properties on matrix measure can be found in [15].

Lemma 2.1: $\mu(A)$ is well defined for any induced norm and has the following properties:

a) $\mu(A) = \lim_{\theta \downarrow 0^+} (\ln ||e^{A\theta}||)/\theta$; $\mu(A) = \min\{\lambda| \, ||e^{At}|| \le e^{\lambda t},\ t \ge 0\}$; $||e^{At}|| \le 1\ (\forall t \ge 0)$ if and only if $\mu(A) \le 0$.

b) For any $\alpha_j \ge 0$ $(1 \le j \le k)$ and any matrices A_j $(1 \le j \le k)$, we have

$$\mu\left(\sum_{j=1}^{k} \alpha_j A_j\right) \le \sum_{j=1}^{k} \alpha_j \mu(A_j).$$

c) For any norm and any A, we have

$$-||A|| \le -\mu(-A) \le \operatorname{Re}\lambda(A) \le \mu(A) \le ||A||.$$

d) Let $|\cdot|$ be a norm on C^n and $P \in C^{n\times n}$ be nonsingular. Let μ_P be the matrix measure induced by the vector norm $|x|_P = |Px|$, then

$$\mu_P(A) = \mu(PAP^{-1}).$$

e) For the 1-norm $|x|_1 = \sum_{i=1}^{n} |x_i|$, the induced matrix measure μ_1 is given by

$$\mu_1(A) = \max_j \left[\operatorname{Re}(a_{jj}) + \sum_{i\ne j} |a_{ij}|\right].$$

For the 2-norm $|x|_2 = (\sum_{i=1}^{n} |x_i|^2)^{1/2}$, the induced matrix measure μ_2 is given by

$$\mu_2(A) = \max_i \left[\frac{\lambda_i(A+A^*)}{2}\right].$$

For the ∞-norm $|x|_\infty = \max_{1\le i\le n} |x_i|$, the induced matrix measure is given by

$$\mu_\infty(A) = \max_i \left[\operatorname{Re}(a_{ii}) + \sum_{j\ne i} |a_{ij}|\right].$$

f) Let H denote a positive definite Hermitian matrix and μ_H denote the matrix measure induced by the vector norm $|x|_H = \sqrt{x^* H x}$, then

$$\mu_H(A) = \tfrac{1}{2}\lambda_{\max}(HAH^{-1} + A^*).$$

g) For any positive numbers $\{r_1, r_2, \cdots, r_n\}$, let $R = \operatorname{diag}\{r_1^{-1}, r_2^{-1}, \cdots, r_n^{-1}\}$, then the matrix measure μ_R^1 induced by the norm $|x|_R = |Rx|_1$ is given by

$$\mu_R^1(A) = \max_j \left[\operatorname{Re}(a_{jj}) + \sum_{i\ne j} \frac{r_i}{r_j} |a_{ij}|\right]$$

and the matrix measure μ_R^∞ induced by the vector norm $|x|_R = |Rx|_\infty$ is given by

$$\mu_R^\infty(A) = \max_i \left[\operatorname{Re}(a_{ii}) + \sum_{j\ne i} \frac{r_i}{r_j} |a_{ij}|\right].$$

h) Let $\mathcal{N}$ be the set of all vector norms on C^n, for any $\rho \in \mathcal{N}$; the corresponding matrix measure is denoted as μ_ρ. Let $\mathcal{P}$ be the set of nonsingular matrices in $C^{n\times n}$ and $\mathcal{H}$ be the set of positive Hermitian matrices in $C^{n\times n}$. Using μ_P^p and μ_H to denote the matrix measures induced by $|x|_P^p = |Px|_p$ $(p = 1, 2, \infty)$ and $|x|_H = \sqrt{x^* H x}$, respectively, then we have

$$\begin{aligned}\max_{1\le i\le n} \operatorname{Re}\lambda_i(A) &= \inf_{\rho\in\mathcal{N}} \mu_\rho(A)\\ &= \inf_{P\in\mathcal{P}} \mu_P^1(A)\\ &= \inf_{P\in\mathcal{P}} \mu_P^2(A)\\ &= \inf_{P\in\mathcal{P}} \mu_P^\infty(A)\\ &= \inf_{H\in\mathcal{H}} \mu_H(A).\end{aligned}$$

i) A is a stable matrix iff there exists a matrix measure μ such that $\mu(A) < 0$.

Proof: The proof is given in the Appendix. □

Although matrix measure is only defined for constant fixed matrices, it can be applied to any matrix, either time-invariant or time-varying, deterministic or stochastic. This is why matrix measure can be used to study the stability of linear time-varying systems or stochastic systems. The key idea is that the estimation of the solution of linear systems can be obtained using the matrix measure technique. The following theorem plays a central role in the subsequent stability analysis.

Theorem 2.2. (Coppel's Inequality): Under fair conditions on $A(t)$ (e.g., piecewise continuous, or integrability condition), the solution of the following linear system:

$$\dot{x}(t) = A(t)x(t) \tag{1}$$

satisfies the inequalities

$$\begin{aligned}&||x(t_0)|| \exp\left\{-\int_{t_0}^{t} \mu[-A(s)]\, ds\right\}\\ &\le ||x(t)||\\ &\le ||x(t_0)|| \exp\left\{\int_{t_0}^{t} \mu[A(s)]\, ds\right\}.\end{aligned}$$

Proof: The proof can be found in either [15] or [21]. □

From Coppel's inequality, we may obtain some easily testable conditions. In what follows, stability always means asymptotic stability in the Lyapunov sense and that $A(t)$ is stable means that (1) with this $A(t)$ is asymptotically stable in the Lyapunov sense.

Corollary 2.3: If there exists a matrix measure μ such that

$$\limsup_{t\to+\infty} \int_{t_0}^{t} \mu[A(s)]\, ds = -\infty$$

then (1) is asymptotically stable. In particular, if there exists a $\alpha > 0$ and a matrix measure μ such that for all sufficiently large t

$$\mu[A(t)] \le -\alpha$$

then (1) is asymptotically stable with the convergence rate at least α.

Proof: The first claim directly follows from Coppel's inequality. We only need to prove the second claim. If for sufficiently large T and for all $t \ge T$, $\mu[A(t)] \le -\alpha$, then for $t \ge T$

$$\begin{aligned} \|x(t)\| &\le \|x(t_0)\| \exp\left\{\int_{t_0}^{t} \mu[A(s)]\, ds\right\} \\ &= \|x(t_0)\| \exp\left\{\int_{t_0}^{T} \mu[A(s)]\, ds\right\} \\ &\quad \cdot \exp\left\{\int_{T}^{t} \mu[A(s)]\, ds\right\} \\ &\le \|x(t_0)\| \exp\left\{\int_{t_0}^{T} \mu[A(s)]\, ds\right\} \\ &\quad \cdot \exp[-\alpha(t-T)] \\ &= M(t_0, T)\|x(t_0)\| \exp[-\alpha(t-t_0)] \end{aligned}$$

where $M(t_0, T) = \exp\{\alpha(T - t_0) + \int_{t_0}^{T} \mu[A(s)]\, ds\}$. Thus (1) is asymptotically stable with convergence rate at least α. This completes the proof. □

III. Stability Analysis of Dynamical Neural Networks

In the last section, we witnessed that matrix measure can be used to estimate the solution of linear differential equations and some testable stability results can be derived. Now we want to extend some of the results to nonlinear systems, in particular, to some classes of dynamical neural networks. Some well-known results will be generalized.

Consider the nonlinear system

$$\dot{x}(t) = f(x, t). \tag{2}$$

Assume that $f(x, t)$ is continuously differentiable with respect to x. Then we have the following theorem.

Theorem 3.1: Let $D_1 f(x, t) = (\partial f/\partial x)(x, t)$ be the Jacobian matrix of $f(x, t)$ with respect to x and assume that $f(0, t) = 0$ for any t, i.e., zero is an equilibrium of (2). Define

$$A(t) = \int_0^1 D_1 f[sx(t), t]\, ds.$$

Then the solution of (2) satisfies the inequalities

$$\begin{aligned} &\|x(t_0)\| \exp\left\{-\int_{t_0}^{t} \mu[-A(s)]\, ds\right\} \\ &\quad \le \|x(t)\| \\ &\quad \le \|x(t_0)\| \exp\left\{\int_{t_0}^{t} \mu[A(s)]\, ds\right\}. \end{aligned}$$

Proof: The proof is similar to the proof of Theorem A in [22] by Desoer and Haneda. Let $g(s) = f[sx(t), t]$, then

$$\begin{aligned} A(t)x(t) &= \int_0^1 D_1 f[sx(t), t]x(t)\, ds \\ &= \int_0^1 \dot{g}(s)\, ds \\ &= g(1) - g(0) \\ &= f[x(t), t] - f(0, t) \\ &= f[x(t), t] \end{aligned}$$

thus the solution of (2) is the same as the solution of

$$\dot{x}(t) = A(t)x(t).$$

From Coppel's inequality, we obtain the result. □

Remark: In this theorem, we assume that the origin is the equilibrium point. This does not lose any generality, as can be seen obviously from the proof.

Although Theorem 3.1 provides the bounds of the solution of a nonlinear differential equation, the bounds involve the solution of the equation so we cannot use this directly. But from this, we can obtain some computable result.

Corollary 3.2: Let

$$\eta(t) = \max_{x\in R^n} \mu\left[\int_0^1 D_1 f(sx, t)\, ds\right]$$

then the solution of (2) satisfies

$$\|x(t)\| \le \|x(t_0)\| \exp\left[\int_{t_0}^{t} \eta(\tau)\, d\tau\right].$$

Corollary 3.3: Let

$$\alpha(t) = \max_{x\in R^n} \max_{0\le s\le 1} \mu[D_1 f(sx, t)]$$

then the solution of (2) satisfies the following inequality:

$$\|x(t)\| \le \exp\left[\int_{t_0}^{t} \alpha(\tau)\, d\tau\right].$$

In particular, if (2) is autonomous, then $\alpha(t) = \alpha$ is constant and

$$\|x(t)\| \le \|x(t_0)\| \exp(\alpha t).$$

Proof: We first prove that

$$\mu[A(t)] \le \int_0^1 \mu\{D_1 f[sx(t), t]\}\, ds.$$

Fix t, let $0 = s_0 < s_1 < \cdots < s_k = 1$ be an equal length partition of $[0, 1]$, then

$$\lim_{k\to\infty} \sum_0^{k-1} (s_{k+1} - s_k) D_1 f[s_k x(t), t] = \int_0^1 D_1 f[sx(t), t]\, ds. \tag{3}$$

Since $s_{k+1} - s_k > 0$, from Lemma 2.1 b), we have

$$\mu\left\{\sum_0^{k-1} (s_{k+1} - s_k) D_1 f[s_k x(t), t]\right\} \le \sum_0^{k-1} (s_{k+1} - s_k)\mu\{D_1 f[s_k x(t), t]\}. \tag{4}$$

Notice that $\mu(A)$ is continuous, using (3) and letting $k \to \infty$ in (4), we obtain the desired result.

Thus, we can obtain

$$\begin{aligned}\mu[A(t)] &\le \int_0^1 \mu\{D_1 f[sx(t), t]\}\, ds \\ &\le \alpha(t).\end{aligned}$$

Applying Theorem 3.1, we can complete the proof.

The autonomous case is trivial from the definition of $\alpha(t)$. □

Corollary 3.3 can be used to study the stability of nonlinear systems.

Corollary 3.4: If there exists a matrix measure μ such that the function $\alpha(t)$ defined above satisfies

$$\limsup_{t\to\infty} \int_{t_0}^t \alpha(\tau)\, d\tau = -\infty$$

then (2) is globally asymptotically stable. In particular, if there exist a matrix measure μ and a positive number m such that for all sufficiently large T

$$\alpha(t) \le -m, \quad \forall t \ge T$$

then (2) is globally exponentially stable with convergence rate at least m.

Proof: Using Corollary 3.3, we can obtain the proof following the same argument for the linear case. □

Remarks:

i) In the definitions of $\eta(t)$ and $\alpha(t)$, if we replace R^n by a suitable subset of R^n containing the equilibrium zero, then the local asymptotic stability can be obtained.

ii) Desoer and Haneda [22] obtained the following proposition: Let $f\colon R^d \longrightarrow R^d$ be continuously differentiable. Assume that there exists a function $m\colon R_+ \longrightarrow R_+$ with $m(\alpha) > 0$ for all nonnegative numbers $\alpha \in R_+$ and $\int_0^\infty m(\alpha)\, d\alpha = +\infty$ such that $\mu[D_1 f(x) \le -m(\alpha) < 0]$ for all $x \in R^d$. Then f is a C^1-diffeomorphism from R^d onto itself. From this result, we can easily show that for any vector $J \in R^d$, $f(x) + J = 0$ always has a unique solution. This means that for autonomous systems, (2) does have a unique equilibrium point when the conditions in Corollary 3.4 hold.

As we have already noticed, matrix measure depends on the choice of the corresponding vector norm. By specifying the vector norm, we can obtain some new test criteria for stability of nonlinear systems. The following classical results can be obtained from this consideration.

First, as an illustration for the choice of appropriate matrix measure, we give different proofs for the following two well-known results.

Corollary 3.5 (Lyapunov [23]): Consider the system

$$\dot{x}(t) = Ax(t) + f(x).$$

If A is stable and $f(x)$ satisfies $\lim_{x\to 0} \|f(x)\|/\|x\| = 0$, then the system is locally asymptotically stable.

Proof: If A is stable, then from i) of Lemma 2.1, there exists a matrix measure μ such that $\mu(A) < 0$. Since μ is continuous, we have

$$\lim_{x\to 0} \mu\left[\frac{\partial f}{\partial x}(x)\right] = 0.$$

Thus, from $\mu[A + (\partial f/\partial x)(x)] \le \mu(A) + \mu[(\partial f/\partial x)(x)]$, we can obtain a neighborhood U around zero for $\varepsilon > 0$ satisfying $\mu(A) + \varepsilon < 0$, such that

$$\max_{x\in U} \mu\left[A + \frac{\partial f}{\partial x}(x)\right] \le \mu(A) + \varepsilon < 0.$$

From the remark following Corollary 3.4, we conclude that the system is locally asymptotically stable. □

Corollary 3.6 (Krasovskii [23]): Let $J = \partial f/\partial x$. If there exists $\delta > 0$ and positive definite constant matrix P such that the eigenvalues $\lambda_1(x, t), \cdots, \lambda_n(x, t)$ of the matrix $J^T P + PJ$ satisfy $\max_i \lambda_i \le -\delta < 0$, then (2) is globally exponentially stable.

Proof: Using norm in f) of Lemma 2.1 with $H = P$, the induced norm $\mu = \mu_P$, then we have

$$\begin{aligned}\mu(J) &= \tfrac{1}{2}\lambda_{\max}(PJP^{-1} + J^T) \\ &= \tfrac{1}{2}\lambda_{\max}[(PJ + J^T P)P^{-1}] \\ &\le \tfrac{1}{2}\lambda_{\max}(PJ + J^T P)\lambda_{\min}(P^{-1}) \\ &\le -\frac{\delta}{2\lambda_{\max}(P)} \\ &< 0.\end{aligned}$$

From Corollary 3.4, Corollary 3.6 is proven. □

Next, we want to specify some other matrix measures to obtain some useful testable conditions for stability. In what follows, we use Y^T to denote the transpose of the matrix Y, and $X < Y$ denotes that $Y - X$ is a positive definite matrix for symmetric matrices X and Y.

Corollary 3.7: System (2) is globally exponentially stable if one of the following conditions holds:

a) There is a positive constant η such that for any $x \in R^n$

$$D_1 f(x, t) + [D_1 f(x, t)]^T < -2\eta I.$$

b) There is a positive number $\eta > 0$ such that the Lyapunov inequality

$$PD_1 f(x, t) + [D_1 f(x, t)]^T P < -\eta I, \quad \forall x \in R^n$$

has a constant positive definite solution.

c) Let $C(x, t) = D_1 f(x, t)$, there is a constant $\eta > 0$ such that

$$c_{ii}(x, t) + \sum_{j \neq i} |c_{ij}(x, t)| < -\eta, \quad \forall i, t \geq 0, x \in R^n$$

or

$$c_{jj}(x, t) + \sum_{i \neq j} |c_{ij}(x, t)| < -\eta, \quad \forall i, t \geq 0, x \in R^n.$$

d) For $\eta > 0$, define a new matrix $M: M_{ii} = -\bar{c}_{ii}$, $M_{ij} = -\bar{c}_{ij}$ $(i \neq j)$, where $\bar{c}_{ij}$ are constants such that $c_{ii}(x, t) \leq \bar{c}_{ii}$ and $|c_{ij}(x, t)| \leq \bar{c}_{ij}$ $(i \neq j)$ for any t and $x \in R^n$. All leading principal minors of matrix M are positive.

Proof: a) can be proven by choosing the 2-norm induced matrix measure in Corollary 3.4 and using Lemma 2.1 e). b) is another version of Corollary 3.6. c) can be proven by choosing the 1-norm or ∞-norm induced matrix measure in Corollary 3.4 and using Lemma 2.1 e). To prove d), we need the following result: [17, Th. 2.5.3] Let $E = (e_{ij})$ be an n by n matrix with $e_{ij} \leq 0$ $(i \neq j)$, then the leading principal minors of E are all positive if and only if the diagonal entries of E are positive and $D^{-1}ED$ is strictly row diagonally dominant for some positive diagonal matrix D, i.e., there exist positive numbers $r_1, \cdots, r_n$ such that

$$\sum_{j \neq i} \frac{r_j}{r_i} |e_{ij}| < e_{ii}, \quad \forall i.$$

According to the definition of M, M is such a matrix if the leading principal minors of M are positive, then there exist positive numbers $r_1, r_2, \cdots, r_n$ which are independent of t and $x \in R^n$ such that

$$\sum_{j \neq i} \frac{r_j}{r_i} \bar{c}_{ij} < -\bar{c}_{ii}, \quad \forall i$$

then there exists a positive number $\eta > 0$ such that

$$\bar{c}_{ii} + \sum_{j \neq i} \frac{r_j}{r_i} \bar{c}_{ij} < -\eta, \quad \forall i.$$

From this, we obtain

$$c_{ii}(x, t) + \sum_{j \neq i} \frac{r_j}{r_i} |c_{ij}(x, t)| < -\eta < 0, \quad \forall i.$$

From Lemma 2.1 g), we have

$$\mu_R^\infty [D_1 f(x, t)] = \mu_R^\infty [C(x, t)] < -\eta$$

where $R = \mathrm{diag}\{r_1, r_2, \cdots, r_n\}$. Now applying Corollary 3.4, we conclude that (2) is globally exponentially stable with convergence rate at least η. This completes the proof. □

In [1, Th. 2] Hirsch proved that the dynamical system $\dot{x} = F(x)$ is globally asymptotically stable if there is a constant $-\eta < 0$ such that $\langle A\xi, \xi \rangle \leq -\eta |\xi|^2$ $(\forall x \in R^n)$. In fact, this condition is exactly the result a) in Corollary 3.7 when the system is autonomous. Also notice that a) is a special case of b) in Corollary 3.7.

We now apply the above results to a class of Hopfield-type dynamical neural networks.

Consider the additive neural networks

$$\dot{x}_i(t) = -\alpha_i x_i(t) + J_i + \sum_{j=1}^{n} w_{ij} g_j[x_j(t)] \qquad i = 1, 2, \cdots, n \tag{5}$$

where $g_j(x_j)$ is a continuously differentiable, monotonically increasing function which represents the firing activity output of the jth neuron; while this usually converts average membrane potential into average firing rate for jth neuron, different neurons may have different firing rates according to their physical functioning. w_{ij} is a constant representing the synoptic strength from neuron j to neuron i, positive values represent excitatory effects which help the firing of ith neuron, while negative values exhibit the inhibitory connections which help to suppress the activity of the ith neuron. J_i is the external stimulus which describes the environmental influence over the ith neuron. α_i is a constant governing the changing rate of the ith neuron. Let $W = (w_{ij})$, $G(x) = [g_1(x_1), \cdots, g_n(x_n)]^T$, $J = (J_1, \cdots, J_n)^T$, $D = \mathrm{diag}\{\alpha_1, \cdots, \alpha_n\}$, $H(x) = (\partial G / \partial x)(x) = \mathrm{diag}\{\dot{g}_1(x_1), \dot{g}_2(x_2), \cdots, \dot{g}_n(x_n)\}$.

This class of dynamical neural networks has been studied by many researchers (see [2] and references therein). A large subset of this class is the well-known Hopfield neural networks [5] that has found practical applications [6]. Kelly [2] studied the global asymptotic stability of this model and obtained a testable condition. As Hirsch [1] noted earlier, if the neural network is globally asymptotically stable for each external input J, then we do not need to specify the initial activity x_i (or the initial pattern) of the neural network, since all trajectories go to the same unique equilibrium, i.e., the global pattern. In this way, we can obtain a mapping from the input pattern J to the global pattern $\bar{x}$, where $\bar{x}$ is the equilibrium. Moreover, we do not need to reset the activations when the external stimuli (or patterns) change. Compare this, for example, with the Kohonen Winner-Take-All networks [24] where resetting must be made when external stimuli change.

Applying the stability results for nonlinear systems we developed above, we can improve some of the previously known results and obtain some new sufficient conditions for global stability of the dynamical neural networks of (5).

Theorem 3.8:

i) If there exists a positive number ε and a matrix measure μ such that $\mu[WH(x) - D] < -\varepsilon$ for any $x \in R^n$, then the neural network (5) is globally exponentially stable.

ii) Assume that $0 \leq \dot{g}_j(x) \leq \beta_j$ $(\forall x \in R^n)$. The neural network (5) is globally exponentially stable if one of the following conditions holds:

a) $\|W\|_2 < (\min_{1 \leq i \leq n} \alpha_i)/(\max_{1 \leq i \leq n} \beta_i)$.

b) Define $y^+ = \max\{y, 0\}$ for any real number y

$$-\alpha_i + w_{ii}^+ \beta_i + \sum_{j \neq i} |w_{ij}| \beta_j < 0, \qquad 1 \leq i \leq n$$

or

$$-\alpha_j + \left[w_{jj} + \sum_{i \neq j} |w_{ij}| \right]^+ \beta_j < 0, \quad 1 \leq j \leq n.$$

c) Let

$$M = \begin{pmatrix} \alpha_1 - w_{ii}^+\beta_1 & -|w_{12}|\beta_2 & \cdots & -|w_{1n}|\beta_n \\ -|w_{21}|\beta_1 & \alpha_2 - w_{22}^+\beta_2 & \cdots & -|w_{2n}|\beta_n \\ \vdots & \vdots & \ddots & \vdots \\ -|w_{n1}|\beta_1 & -|w_{n2}|\beta_2 & \cdots & \alpha_n - w_{nn}^+\beta_n \end{pmatrix}.$$

The leading principal minors of matrix M are positive.

d) For any $i \in \{1, 2, \cdots, n\}$

$$-\alpha_i + w_{ii}^+\beta_i + \tfrac{1}{2}\sum_{j\neq i}(|w_{ij}|\beta_j + |w_{ji}|\beta_i) < 0.$$

e) Let M_s denote the matrix shown at the bottom of the page. The leading principal minors of matrix M_s are positive, or equivalently, M_s is positive definite.

Proof:

i) Let $f(x) = -D + J + WG(x)$, then (5) becomes the differential equation $\dot{x} = f(x)$. Moreover, we have $D_1 f(x) = -D + WH(x)$. Thus, we have

$$\mu[D_1 f(x)] \leq -\varepsilon.$$

Applying Corollary 3.4, we can easily complete the proof of i).

ii) a) For the 2-norm, we can easily obtain

$$\begin{aligned} \mu_2(-D) &= \min_{1\leq i\leq n} \alpha_i \\ ||H(x)||_2 &= \max_{1\leq i\leq n} |\dot{g}(x_i)| \\ &\leq \max_{1\leq i\leq n} \beta_i. \end{aligned}$$

From the given condition, there exists a $\varepsilon > 0$ such that

$$||W||_2 < \frac{\left(\min\limits_{1\leq i\leq n} \alpha_i\right) - \varepsilon}{\max\limits_{1\leq i\leq n} \beta_i}.$$

Using this and Lemma 2.1 c), we can easily obtain

$$\begin{aligned} \mu_2[WH(x)] &\leq ||WH(x)||_2 \\ &\leq ||W||_2 ||H(x)||_2 \\ &\leq -\mu_2(-D) - \varepsilon. \end{aligned}$$

From i), we prove ii-a).

b) Using notation in Corollary 3.7, we have for (5)

$$\begin{aligned} c_{ii} &= -\alpha_i + w_{ii}\dot{g}_i(x_i) \\ c_{ij} &= w_{ij}\dot{g}_j(x_j) \quad (i \neq j). \end{aligned}$$

From the given condition, there exists a $\eta > 0$ such that

$$-\alpha_i + w_{ii}^+\beta_i + \sum_{j\neq i} |w_{ij}|\beta_j < -\eta, \quad 1 \leq i \leq n.$$

Thus, we obtain

$$\begin{aligned} & c_{ii}(x) + \sum_{j\neq i} |c_{ij}(x)| \\ &= -\alpha_i + w_{ii}\dot{g}_i(x_i) + \sum_{j\neq i} |w_{ij}|\, |\dot{g}_j(x_j)| \\ &\leq -\alpha_i + w_{ii}^+\beta_i + \sum_{j\neq i} |w_{ij}|\beta_j \\ &\leq -\eta. \end{aligned}$$

Applying Corollary 3.7 c), we can easily complete the proof of the first part of b). The second part of b) can be proven in a similar fashion.

c) The proof follows from a modification of the proof of Corollary 3.7 d). If the given condition holds, then there exists positive numbers $r_1, \cdots, r_n$ such that

$$-\alpha_i + w_{ii}^+ + \sum_{j\neq i} \frac{r_j}{r_i} |w_{ij}|\beta_j < 0.$$

From this, there exists a positive number $\eta > 0$ such that

$$-\alpha_i + w_{ii}^+ + \sum_{j\neq i} \frac{r_j}{r_i} |w_{ij}|\beta_j < -\eta.$$

From this, the proof can be completed very easily.

d) First, we notice: For any matrix A, $(A + A^T)/2 < -\eta I$ if and only if $\lambda_{\max}(A + A^T)/2 < -\eta$. Since eigenvalues of any symmetric matrix are real, from Lemma 2.1 c) we have $\lambda_{\max}(A+A^T)/2 < \mu_1[(A+A^T)/2]$. Using a similar argument as in the proof of b) and Corollary 3.7, we can easily complete the proof.

e) Similar to d), we can substitute $D_1 f(x)$ with the matrix $D_1 f(x) + [D_1 f(x)]^T$ in the proof of c), and the proof follows very easily. □

Remarks:

a) When the decaying factors α_i of all neurons are the same, the network (5) is reduced to the case Kelly [2] studied; ii-a) is reduced to Kelly's [2] sufficient condition for global asymptotic stability. When $\beta_1 = \beta_2 = \cdots = \beta_n$, ii-d) is reduced to Hirsch [1, Th. 3] (There is a small mistake there, the w_{ii} should be replaced by w_{ii}^+, since w_{ii} may be a negative number.) From the proof of ii-c)

$$\begin{pmatrix} \alpha_1 - w_{ii}^+\beta_1 & -\dfrac{|w_{12}|\beta_2 + |w_{21}|\beta_1}{2} & \cdots & -\dfrac{|w_{1n}|\beta_n + |w_{n1}|\beta_1}{2} \\ -\dfrac{|w_{21}|\beta_1 + |w_{12}|\beta_2}{2} & \alpha_2 - w_{22}^+\beta_2 & \cdots & -\dfrac{|w_{2n}|\beta_n + |w_{n2}|\beta_2}{2} \\ \vdots & \vdots & \ddots & \vdots \\ -\dfrac{|w_{n1}|\beta_1 + |w_{1n}|\beta_n}{2} & -\dfrac{|w_{n2}|\beta_2 + |w_{2n}|\beta_n}{2} & \cdots & \alpha_n - w_{nn}^+\beta_n \end{pmatrix}$$

and e), we notice that we have used the Lemma 2.1 g) which has Lemma 2.1 e) as a special cases for 1-norm and ∞-norm, therefore, ii-c) and e) are better results over ii-b) and d), respectively.

b) From the Remark ii) following Corollary 3.4, we can easily observe that the conditions in Theorem 3.8 also guarantee that the neural network (5) does have a unique equilibrium point.

c) From the stability criteria in Theorem 3.8, we see that the global stability does not depend on the external stimuli vector J. However, the external input J does affect the location of the equilibrium point.

d) From ii-a), b), or d), it is easy to observe that given decay factors α_i and synoptic connection strength w_{ij}, we can always find neural activations $g_j(x_j)$ such that the neural network (5) is globally asymptotically stable. In fact, we can choose β_i sufficiently small so that the conditions in Theorem 3.8 can be satisfied. The smallness of such β_i indicates the flatness of the neural activation, and the neural activity does not saturate easily.

e) If $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha$, all the criteria in Theorem 3.8 can be simplified. For example, the neural network (5) is globally exponentially stable if there exists an $\eta > 0$ such that $\mu[WH(x)] \le \alpha - \eta$.

By specifying the matrix measure as in Corollary 3.7, we can easily obtain the following result.

Theorem 3.9: The neural network (5) is globally exponentially stable if one of the following conditions holds:

a) There exists an $\eta > 0$ and positive definite matrix P such that

$$-PD - DP + WH(x)P + PH(x)W^T < -\eta I, \quad \forall x \in R^n.$$

b) There exists an $\eta > 0$ such that for any $x \in R^n$ and $i \in \{1, 2, \cdots, n\}$

$$-\alpha_i + w_{ii}\dot{g}_i(x_i) + \sum_{j \ne i} |w_{ij}|\dot{g}_j(x_j) < -\eta.$$

c) There exists an $\eta > 0$ such that for any $x \in R^n$ and $j \in \{1, 2, \cdots, n\}$

$$-\alpha_j + \left[w_{jj} + \sum_{i \ne j} |w_{ij}| \right] \dot{g}_j(x_j) < -\eta.$$

d) There exist positive numbers $r_1, r_2, \cdots, r_n, \eta$ such that for any $x \in R^n$ and $i \in \{1, 2, \cdots, n\}$

$$-\alpha_i + w_{ii}\dot{g}_i(x_i) + \sum_{j \ne i} \frac{r_j}{r_i} |w_{ij}|\dot{g}_j(x_j) < -\eta.$$

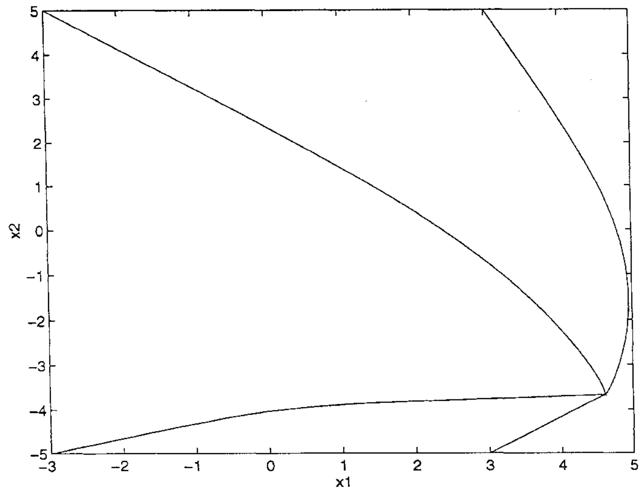

Fig. 1. Globally stable network phase plot with different initial conditions.

Proof: The proof follows directly from Corollary 3.7 or Lemma 2.1. □

Remark: A necessary condition for d) to hold is that the matrix shown at the bottom of the page has positive leading principal minors for all $x \in R^n$.

Example 1: Let $h(x) = (e^x - e^{-x})/(e^x + e^{-x})$ and $g_i(x) = h(a_i x)$ $(i = 1, 2, \cdots, n)$, where $a_1, \cdots, a_n$ are positive numbers. It is easy to show that

$$\begin{aligned} \dot{g}_i(x) &= a_i[1 - h^2(a_i x)] \\ \beta_i &= \max_{x \in R} = \dot{g}_i(0) \\ &= a_i > 0. \end{aligned}$$

Let $n = 2$, $\alpha_1 = \alpha_2 = 1$, $w_{11} = 0.5$, $w_{22} = 0.2$, $w_{12} = 1$, $w_{21} = -0.5$. Obviously, the condition in Theorem 3.8 ii-b) holds. Therefore, the two-neuron network (5)

$$\begin{aligned} \dot{x}_1 &= -x_1 + 0.5h(x_1) + h(0.4x_2) + J_1 \\ \dot{x}_2 &= -x_2 - 0.5h(x_1) + 0.2h(0.4x_2) + J_2 \end{aligned}$$

is globally exponentially stable for any input stimuli J_1 and J_2. Let $J_1 = 5$ and $J_2 = -3$. The phase plots for this network with four initial conditions $(3, 5)$, $(-3, 5)$, $(3, -5)$, and $(-3, -5)$ are shown in Fig. 1. It shows that the neural network does converge to the unique equilibrium point $(4.6, -3.68)$.

Local stability analysis has been done by many researchers in the last few years [3], [4], [11], [12], and many sufficient conditions for local stability, in particular, local exponential stability of Hopfield-type neural networks have been obtained. For (5), there may be many equilibrium points which represent local patterns. In an associative memory, we may wish to

$$M(x) = \begin{pmatrix} \alpha_1 - w_{11}\dot{g}_1(x_1) & -|w_{12}|\dot{g}_2(x_2) & \cdots & -|w_{1n}|\dot{g}_n(x_n) \\ -|w_{21}|\dot{g}_1(x_1) & \alpha_2 - w_{22}\dot{g}_2(x_2) & \cdots & -|w_{2n}|\dot{g}_n(x_n) \\ \vdots & \vdots & \ddots & \vdots \\ -|w_{n1}|\dot{g}_1(x_1) & -|w_{n2}|\dot{g}_2(x_2) & \cdots & \alpha_n - w_{ii}\dot{g}_n(x_n) \end{pmatrix}$$

design (5) such that it has as many stable equilibrium points as possible to increase the capacity of the neural network ([3] and references therein). Many results in the literature for local stability are in fact for local exponential stability although Lyapunov's second method had been used. Next, we will show that most of the sufficient conditions for local (exponential) stability of dynamical neural networks can be obtained by the linearized systems with the help of matrix measure technique.

The following result shows that local exponential stability of a nonlinear system is equivalent to the stability of its linearized system.

Lemma 3.1. [25], [26]: Suppose that $f(x)$ is continuously differentiable in the neighborhood of the origin and $f(0) = 0$. Define $A = (\partial f/\partial x)(x)|_{x=0}$. Then the nonlinear system $\dot{x}(t) = f[x(t)]$ is locally exponentially stable at the origin if and only if the linearized system $\dot{z}(t) = Ax(t)$ is stable, i.e., the matrix A is Hurwitz stable (that is, the eigenvalues of A have negative real parts). □

Now, we use the linearized system of (5) to obtain some sufficient conditions for the local exponential stability of the equilibrium points.

Theorem 3.10: Let x^* be any equilibrium point of (5).

1) System (5) is locally exponentially stable at x^* if and only if there exists a matrix measure μ such that $\mu[-D+WH(x^*)] < 0$.
2) System (5) is locally exponentially stable at x^* if one of the following conditions holds:
   a) There exists a matrix measure μ such that $\mu[WH(x^*)] < -\mu(-D)$,
   b) $\lambda_{\max}[WH(x^*) + H(x^*)W^T] < \min_{1\le i\le n} \alpha_i$;
   c) Any condition in Theorem 3.8 ii) with β_j replaced by $\dot{g}_j(x_j^*)$ $(j = 1, 2, \cdots, n)$ and w_{ii}^+ replaced by w_{ii} $(i = 1, 2, \cdots, n)$,
   d) There exist a matrix measure μ and a nonsingular matrix P such that $\mu\{P[-D+WH(x^*)]P^{-1}\} < 0$,
   e) $\dot{g}_i(x_i^*) > 0$ and $\lambda_{\max}(W + W^T) < 2\alpha_i/\dot{g}_i(x_i^*)$ $(i = 1, 2, \cdots, n)$,
   f) $\dot{g}_i(x_i^*) > 0$ and $w_{ii} + \frac{1}{2}\sum_{j\ne i} |w_{ij} + w_{ji}| < \alpha_i/\dot{g}_i(x_i^*)$ $(i = 1, 2, \cdots, n)$,
   g) Let $M = (m_{ij})$ where

$$m_{ij} = \begin{cases} \alpha_i - w_{ii}\dot{g}_i(x_i^*), & i = j \\ -\dfrac{|w_{ij}\dot{g}_j(x_j^*) + w_{ji}\dot{g}_i(x_i^*)|}{2}, & i \ne j. \end{cases}$$

   The principal minors of M are positive.

Proof:

1) From Lemma 3.1, x^* is locally exponentially stable if and only if the linearized system around x^* is stable. The system matrix of the linearized system is $-D + WH(x^*)$, from Lemma 2.1 i), and $-D + MH(x^*)$ is stable if and only if there exists a matrix measure μ such that $\mu[-D + WH(x^*)] < 0$. This proves 1).
2) a) This can be proven easily by 1) and Lemma 2.1 b).
   b) In a), choosing the matrix measure induced by the 2-norm, from Lemma 2.1 e), we can complete the proof of b).
   c) The proof of c) can be easily replicated from the proof of Theorem 3.8.
   d) This is straightforward from Lemma 2.1 d) and 1).
   e) Since $H(x^*)$ is a diagonal matrix with positive diagonal elements, we can formally obtain its square root $P = \sqrt{H(x^*)}$. Using the 2-norm, let a new norm be defined as $||x|| = ||Px||_2$ with the induced matrix measure denoted by μ. Let $A = -D + WH(x^*)$. Then we have

$$\begin{aligned}
&\mu(A)\\
&= \mu_2(PAP^{-1})\\
&= \tfrac{1}{2}\lambda_{\max}(PAP^{-1} + P^{-1}A^TP)\\
&= \tfrac{1}{2}\lambda_{\max}(P[AP^{-2} + P^{-2}A^T]P)\\
&\le \tfrac{1}{2}\lambda_{\max}^2(P)\lambda_{\max}[AH^{-1}(x^*) + H^{-1}(x^*)A^T]\\
&= \tfrac{1}{2}\lambda_{\max}^2(P)\lambda_{\max}\{[-D + WH(x^*)]H^{-1}(x^*)\\
&\quad + H^{-1}(x^*)[-D + WH(x^*)]^T\}\\
&= \tfrac{1}{2}\lambda_{\max}^2(P)\{\lambda_{\max}[-2DH^{-1}(x^*)]\\
&\quad + \lambda_{\max}(W + W^T)\}\\
&= \tfrac{1}{2}\lambda_{\max}^2(P)\{2\lambda_{\max}[-DH^{-1}(x^*)] + 2\mu_2(W)\}\\
&= \lambda_{\max}^2(P)\left\{-\min\left[\frac{\alpha_i}{\dot{g}_i(x_i^*)}\right] + \mu_2(W)\right\} < 0.
\end{aligned}$$

   From 1), we conclude that x^* is locally exponentially stable.
   f) From the proof of e), we notice (or directly from the Lyapunov theory) that x^* is locally exponentially stable if there exists a positive definite matrix P such that $AP^{-2} + P^{-2}A^T$ is negative definite, or $\mu_\infty(AP^{-2} + P^{-2}A^T) < 0$ (if X is a symmetric matrix, then X is negative definite if and only if X is stable). Choosing P as in the proof of e), we obtain

$$\begin{aligned}
&\mu_\infty(AP^{-2} + P^{-2}A^T)\\
&\quad = \mu_\infty[-DH^{-1}(x^*) + W + W^T]\\
&\quad = 2\max_{1\le i\le n}\left\{-\alpha_i g^{-1}(x_i^*)w_{ii} + \frac{1}{2}\sum_{j\ne i}|w_{ij} + w_{ji}|\right\}\\
&\quad < 0.
\end{aligned}$$

   This completes the proof of f).
   g) As in the proof of f), X^* is locally exponentially stable if $\mu_R^\infty[(A + A^T)/2] < 0$. Let $X = (x_{ij}) = (A + A^T)/2$, then $m_{ij} = |x_{ij}|$. Following the procedure as in the proof of Theorem 3.8 d) or e), we can complete the proof of g). □

Remarks:

1) In c), the parallel condition in Theorem 3.8 ii-b) gives the sufficient condition for local exponential stability

$$-\alpha_i + w_{ii}\dot{g}_i(x_i^*) + \sum_{j\ne i}|w_{ij}|\dot{g}_j(x_j^*) < 0, \quad 1 \le i \le n.$$

This is exactly the condition obtained in [3]. Notice that the sum of the first two terms is negative which is automatically implied by the inequality. Thus, [3, inequalities (11) and (12)] can be combined in this way.

2) In Yang and Dillon [7], the conditions e) and f) were derived under [7, condition (14)]: the second derivatives of the activation functions $g_j(x_j)$ are bounded. Obviously, this condition is not needed.
3) It is easy to show that g) implies f). The computation of g) is obviously much more involved. In fact, g) can be regarded as a generalization of the M-matrix result in [11]. We will illustrate this in the next example.

Example 2 [7]: Let $W = \left(\begin{smallmatrix} a & -2 \\ 2 & a \end{smallmatrix}\right)$, $\alpha_1 = \alpha_2 = \alpha$ and $\dot{g}_1(x_1^*) = \dot{g}_2(x_2^*) = \sigma$, also $0 < \alpha/\sigma - 2 < a < \alpha/\sigma$. In [7], it was shown that the M–matrix result in [11] fails to test the stability of this network. Obviously, e) and f) can be used to ensure that x^* is locally exponentially stable. In g), the matrix M is given by $M = \left(\begin{smallmatrix} \alpha - a\sigma & 0 \\ 0 & \alpha - a\sigma \end{smallmatrix}\right)$ whose principal minors are obviously positive. Therefore, g) can also ensure the local exponential stability of x^*.

Example 3: We still use the same activation matrix W. Now we are interested in the global stability of this neural network (5). Let σ be the maximum value of the slopes of the activation functions, i.e., $\sigma = \max_{i=1,2} \max_{x \in R} \dot{g}_i(x)$. Let $\alpha_1 = \alpha_2 = \alpha$. We assume now that $0 < \alpha/\sigma - 2 < a < \alpha/\sigma - 1$. Similar discussions as in [7] reveal that all previously known results fail to test the stability of the neural network (5). We want to apply Theorem 3.9 a) to study the global exponential stability of (5). Let $\lambda_i(x_i) = \dot{g}_i(x_i)$ $(i = 1, 2)$. It is obvious that $0 \le \lambda_i(x_i) \le 1$ $(i = 1, 2)$. Choosing $P = \operatorname{diag}\{\sigma_1^{-1}, \sigma_2^{-1}\}$, using $\Longleftrightarrow$ to denote the relationship "equivalent to" and $\Longleftarrow$ "implied," we have ($\eta > 0$ is sufficiently small number)

$$
\begin{aligned}
&-PD - DP + WH(x)P + PH(x)W^T < -2\eta I \\
&\Longleftrightarrow \begin{bmatrix} -\dfrac{\alpha}{\sigma} + \eta + a\lambda_1(x_1) & \lambda_1(x_1) - \lambda_2(x_2) \\ \lambda_1(x_1) - \lambda_2(x_2) & -\dfrac{\alpha}{\sigma} + \eta + a\lambda_2(x_2) \end{bmatrix} \\
&\quad < 0 \\
&\Longleftarrow \mu_\infty \begin{bmatrix} -\dfrac{\alpha}{\sigma} + \eta + a\lambda_1(x_1) & \lambda_1(x_1) - \lambda_2(x_2) \\ \lambda_1(x_1) - \lambda_2(x_2) & -\dfrac{\alpha}{\sigma} + \eta + a\lambda_2(x_2) \end{bmatrix} \\
&\quad < 0 \\
&\Longleftarrow -\frac{\alpha}{\sigma} + \eta + a\lambda_i(x_i) \\
&\qquad + |\lambda_1(x_1) - \lambda_2(x_2)| < 0 \quad (i = 1, 2) \\
&\Longleftarrow -\frac{\alpha}{\sigma} + \eta + a^+ + 1 < 0.
\end{aligned}
$$

In the last implication, we have used $0 \le \lambda_i(x_i) \le 1$ and $|\lambda_1(x_1) - \lambda_2(x_2)| \le 1$ which is obvious from the definition of $\lambda_i(x_i)$. From the assumption for sufficiently small $\eta > 0$, $-\alpha/\sigma + \eta + a^+ + 1 < 0$, so the condition in Theorem 3.9 a) is satisfied. From Theorem 3.9 a), we conclude that (5) is globally exponentially stable.

The results in [7] for stability are for local exponential stability only. There is a gap in the results (Theorems 1–4 or their corollaries). The results for the existence and uniqueness of an equilibrium point of a neural network are global properties, while the stability results in [7] are local properties. A question is whether or not a neural network with a unique equilibrium point which is locally exponentially stable is also globally exponentially stable. This issue is not resolved in the paper [7]. For general nonlinear dynamical systems, this is not true, i.e., although a nonlinear system has a unique equilibrium point which is also locally exponentially stable and its trajectory is bounded, the system may not be globally stable. For example, consider the system

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 + (-1 + x_1^2 + x_2^2)(2 - x_1^2 - x_2^2)x_2.
\end{aligned}
$$

Obviously, the system has a unique equilibrium point $(0, 0)^T$, the linearized system around the origin is stable; hence the origin is locally exponentially stable. Consider the Lyapunov function $V(x_1, x_2) = x_1^2 + x_2^2$, we have $\dot{V}(x_1, x_2) = (-1 + x_1^2 + x_2^2)(2 - x_1^2 - x_2^2)x_2^2$. From this, we can conclude that the trajectory is bounded. There are two limit circles for this system: $C_1 = \{x | x_1^2 + x_2^2 = 1\}$ and $C_2 = \{x | x_1^2 + x_2^2 = 2\}$. The first limit circle C_1 is unstable, and the second limit circle C_2 is asymptotically stable. We believe that certain types of (5) will also have similar dynamical behavior. This will be investigated in the future.

IV. Conclusions

In this paper, we systematically study the global as well as the local stability of nonlinear dynamical systems and a class of dynamical neural networks. Based on the fact that the local exponential stability of a nonlinear system is equivalent to the stability of the linearized system, we can say that the local exponential stability of a dynamical neural network can be easily studied. The above equivalence relationship is no longer valid for asymptotic stability, so the local asymptotic stability of a nonlinear system cannot be completely retrieved by its linearized system. However, to guarantee the speed of convergence, in practice, the exponential stability is more desirable. Hence, the linearized system should be enough for certain practical applications. This idea should be very helpful for local design of dynamical neural networks. For example, for local associative memory networks, we only need to design the interconnection matrix, the slopes of the activation functions, and decaying factors to satisfy the stability conditions of the linearized system [3]. However, as we noticed before, the global stability of a nonlinear dynamical system cannot be derived in any sense from the linearized system, even in the case when the unique equilibrium point is locally exponentially stable. The problem remains whether this is true for the Hopfield-type dynamical neural networks. What is the relationship between the local stability of the unique equilibrium point and the global stability for a Hopfield-type neural network? Yang and Dillon [7] investigated the possible oscillations of very specific structured neural networks; it is worthwhile to investigate the rich dynamics such as oscillations and chaos for the dynamical neural networks (5). These related issues will be investigated in the future.

APPENDIX

Proof of Lemma 2.1: a) The second claim can be proven directly from the first claim. The first claim can be proven by Taylor expansion.

The proof of b)–e) can be found in [15].

f) Since H is positive definite, there exists a nonsingular matrix P so that $H = P^*P$. Then $|x|_H = |Px|_2$, using d) and e), we have

$$\begin{aligned}\mu_H(A) &= \mu_2(PAP^{-1})\\ &= \tfrac{1}{2}\lambda_{\max}(PAP^{-1} + P^{-*}A^*P^*)\\ &= \tfrac{1}{2}\lambda_{\max}(HAH^{-1} + A^*).\end{aligned}$$

g) Since

$$RAR^{-1} = \begin{pmatrix} a_{11} & \frac{r_2}{r_1}a_{12} & \cdots & \frac{r_n}{r_1}a_{1n} \\ \frac{r_1}{r_2}a_{21} & a_{22} & \cdots & \frac{r_n}{r_2}a_{2n} \\ \vdots & \vdots & & \vdots \\ \frac{r_1}{r_n}a_{n1} & \frac{r_2}{r_n}a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

using d) and e), we obtain the proof of g).

h) In fact, we have proven in [18] that $\max_{1\le i\le n} \operatorname{Re}\lambda_i(A) = \inf_{P\in\mathcal{P}} \mu_P^2(A)$ for the 2-norm, i.e., $p = 2$ case. From this, $\max_{1\le i\le n} \operatorname{Re}\lambda_i(A) = \inf_{\rho\in\mathcal{N}} \mu_\rho(A)$ is straightforward.

For any A, there exists nonsingular matrix T such that $TAT^{-1} = J$ where J is Jordan form, let $R = \operatorname{diag}\{r_1, r_2, \cdots, r_n\}$, where $r_1, r_2, \cdots, r_n$ are positive numbers to be determined, then let $P = RT \in \mathcal{P}$, and we have

$$PAP^{-1} = RTAT^{-1}R^{-1} = \begin{pmatrix} \lambda_1 & \frac{r_1}{r_2}e_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \frac{r_2}{r_3}e_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ddots & \frac{r_{n-1}}{r_n}e_{n-1} \\ 0 & 0 & 0 & 0 & \lambda_n \end{pmatrix}$$

where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of A and $e_j = 0$ or 1 $(1 \le j \le n-1)$.

For $p = 1$, using e), we have

$$\begin{aligned}\mu_P^1(A) &= \mu_1(PAP^{-1})\\ &= \max_j \left[\operatorname{Re}(\lambda_j) + \frac{r_j}{r_{j+1}}e_j\right].\end{aligned}$$

Thus, we choose R in the way that $r_j/r_{j+1} < \varepsilon$, then we have $\mu_P^1(A) < \max_j \operatorname{Re}\lambda_j(A) + \varepsilon$. This proves the case where $p = 1$.

In a similar fashion, we can prove the case where $p = \infty$.

Let $H = (RT)^*RT$, then $H \in \mathcal{H}$, and

$$\begin{aligned}\mu_H(A) &= \tfrac{1}{2}\lambda_{\max}(HAH^{-1} + A^*)\\ &= \tfrac{1}{2}\lambda_{\max}[RTAT^{-1}R^{-1} + (RTAT^{-1}R^{-1})^*]\\ &\le \max_j \left[\operatorname{Re}(\lambda_j) + \frac{1}{2}\left(\frac{r_{j-1}}{r_j} + \frac{r_j}{r_{j+1}}\right)\right].\end{aligned}$$

Thus using a similar choice to R, we can obtain $\mu_H(A) \le \max_j \operatorname{Re}\lambda_j(A) + \varepsilon$; this proves the last equality in h). Summarizing the above, we complete the proof of h).

i) This is a direct result of h). □

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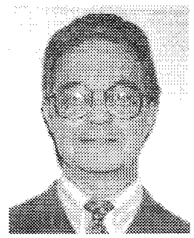

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