New Estimates for Solutions of Lyapunov Equations

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***Abstract*— In this paper, new results for estimating the solution of differential and algebraic Lyapunov matrix equations are obtained, and some of the well-known results are generalized.**

***Index Terms*—Inequality, Lyapunov equations, matrix measure, trace.**

I. Introduction

The Lyapunov matrix equation is very important for stability analysis in control theory. Although the exact solution of the Lyapunov equation can be found numerically, the computational burden increases with the dimension of the system matrices. For some applications such as stability analysis, it is often not necessary to know the exact solution because an estimate of the solution is sufficient. Also, if the parameters in the system matrices are uncertain, it is not possible to obtain the exact solution for robust stability analysis; therefore, it is necessary to find a reasonable estimate for the solution of the Lyapunov equation to obtain some robust stability results. In [7], we have used such an approach to study robust stability and performance analysis for uncertain stochastic systems.

The estimation problem for the solution of the Riccati and Lyapunov matrix equations has attracted considerable attention in the past two decades. Mori and Derese [1] gave a very good summary on this topic. In most works, the lower and upper bounds are for the following quantities: the largest eigenvalue, the trace, the determinant, the partial summation of eigenvalues, the partial product of eigenvalues, and the solution itself. There are plenty of results for obtaining the lower bounds for these quantities; however, in practical applications, especially for stability analysis, the upper bounds for the trace and the largest eigenvalue are desirable [7]. Recently, Komaroff [2], [3], [9] used majorization techniques to obtain some very excellent estimates for the partial summation and partial product of the solution of Lyapunov matrix equations. Mrabti and Hmamed [8] presented a unified approach using the delta operator technique to obtain lower bound estimates for the solution of both continuous-time and discrete-time Lyapunov matrix equations.

In most cases, the bounds that have been obtained are the best possible under some, unfortunately, restrictive assumptions. For example, for most of these bounds, the common assumption is that $A + A'$ is negative definite. This is obviously restrictive, because the stability of A does not guarantee this assumption. In this paper, we will remove this assumption and provide some general estimates for continuous-time Lyapunov matrix differential and algebraic equations. Because of their importance in robust stability and performance analysis, special attention will be given to upper bound estimates for the trace.

II. Notations and Preliminaries

In what follows, we will use the following notations: A is a real $n \times n$ matrix, A' denotes the matrix transpose, $\mathrm{tr}\,(A)$ is the trace of $A, \lambda_i(A)$ is an eigenvalue of $A, (\lambda_i(A))$ are arranged in nonascending order when they are real, i.e., $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A), \Re\lambda_i(A)$ denotes the real part of $\lambda_i(A)$, and $(\Re\lambda_i(A))$ are arranged in nonascending order, i.e., $\Re\lambda_1(A) \geq \Re\lambda_2(A) \geq \cdots \geq \Re\lambda_n(A)$. Let $\mu(\cdot)$ denote the matrix measure induced by some vector or matrix norm and defined by the formula

$$\mu(A) = \lim_{\theta \downarrow 0^+} \frac{||I + \theta A|| - 1}{\theta}.$$

The matrix measure induced by the 2-norm (i.e., the Euclidean norm) is denoted by $\mu_2(A)$ and $\mu_2(A) = \frac{1}{2}\,\lambda_1(A + A')$. Properties of the matrix measure can be found in [11] and [12].

Lemma 2.1: For any matrix A and any symmetric matrix B[6], let $\overline{A} = (A + A')/2$, then we have

$$\begin{aligned}\lambda_n(\overline{A})\,\mathrm{tr}\,(B) - \lambda_n(B)(n\lambda_n(\overline{A}) - \mathrm{tr}\,(A)) \\ \leq \mathrm{tr}\,(AB) \leq \lambda_1(\overline{A})\,\mathrm{tr}\,(B) - \lambda_n(B) \\ \cdot (n\lambda_1(\overline{A}) - \mathrm{tr}\,(A)).\end{aligned}$$

In particular, for any positive semidefinite matrix B we have

$$\lambda_n(\overline{A})\,\mathrm{tr}\,(B) \leq \mathrm{tr}\,(AB) \leq \lambda_1(\overline{A})\,\mathrm{tr}\,(B). \qquad \square$$

Lemma 2.2: Let $\mu_F(\cdot)$ denote the matrix induced by the vector norm $||x|| = \sqrt{x'Fx}$, where F is a positive definite matrix. Let $\mathcal{N}$ denote the set of positive definite matrices. Then for any matrix A, we have

$$\Re\lambda_1(A) = \max_{1 \leq i \leq n} \Re\lambda_i(A) = \inf_{F \in \mathcal{N}} \mu_F(A).$$

Moreover, the matrix measure $\mu_F(A)$ is given by

$$\mu_F(A) = \tfrac{1}{2}\,\lambda_1(FAF^{-1} + A') = \mu_2(F^{1/2}AF^{-1/2})$$

where the Euclidean norm-induced matrix measure is given by $\mu_2(A) = \frac{1}{2}\,\lambda_1(A + A')$.

Proof: The proof of this result is similar to [11, Th. 4]. $\square$

Lemma 2.3 [10, p. 515]: For any matrix A, we have

$$\mathrm{tr}\,(e^A e^{A'}) \leq \mathrm{tr}\,(e^{A+A'}). \qquad \square$$

III. Main Results

Consider the differential Lyapunov matrix equation

$$\dot{P}(t) = A'P(t) + P(t)A + Q, \quad P_0 = P(t_0) \tag{1}$$

and the algebraic Lyapunov matrix equation

$$A'P + PA + Q = 0 \tag{2}$$

where Q is a constant positive semidefinite matrix and A is a constant (Hurwitz) stable real matrix. The main objective of this paper is to find estimates for the positive semidefinite solution matrices $P(t)$ and P for (1) and (2), respectively.

We first give an upper bound for the trace of the solution $P(t)$ of the differential Lyapunov matrix equation (1).

Theorem 3.1: Suppose that the real matrix A is stable and $A + A'$ is nonsingular, then we have

$$\begin{aligned}\mathrm{tr}\,(P(t)) \leq \lambda_1(P(t_0))\,\mathrm{tr}\,(e^{(A+A')(t-t_0)}) - \lambda_1(Q) \\ \cdot \mathrm{tr}\,((A + A')^{-1}) + \lambda_1(Q) \sum_{i=1}^{n} \frac{e^{\lambda_i(A+A')(t-t_0)}}{\lambda_i(A + A')}.\end{aligned}$$

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Proof: The solution of (1) can be expressed as

$$P(t) = e^{A'(t-t_0)}P(t_0)e^{A(t-t_0)} + \int_{t_0}^{t} e^{A'(t-s)}Qe^{A(t-s)}\,ds.$$

From this, we can obtain

$$\begin{aligned}\operatorname{tr}(P(t)) &= \operatorname{tr}((e^{A'(t-t_0)}P(t_0)e^{A(t-t_0)}) \\ &\quad + \int_{t_0}^{t} \operatorname{tr}(e^{A'(t-s)}Qe^{A(t-s)})\,ds \\ &= \operatorname{tr}(P(t_0)e^{A(t-t_0)}e^{A'(t-t_0)}) \\ &\quad + \int_{t_0}^{t} \operatorname{tr}(Qe^{A(t-s)}e^{A'(t-s)})\,ds.\end{aligned}$$

Applying Lemma 2.1 and Lemma 2.3 to the right side of the above inequality, we have

$$\begin{aligned}\operatorname{tr}(P(t)) &\le \lambda_1(P(t_0))\operatorname{tr}(e^{(A+A')(t-t_0)}) + \lambda_1(Q) \\ &\quad \cdot \int_{t_0}^{t}\operatorname{tr}(e^{(A+A')(t-s)})\,ds \\ &= \lambda_1(P(t_0))\operatorname{tr}(e^{(A+A')(t-t_0)}) + \lambda_1(Q) \\ &\quad \cdot \int_{t_0}^{t}\sum_{i=1}^{n} e^{\lambda_i(A+A')(t-s)}\,ds \\ &= \lambda_1(P(t_0))\operatorname{tr}(e^{(A+A')(t-t_0)}) + \lambda_1(Q) \\ &\quad \cdot \sum_{i=1}^{n}\left(-\frac{e^{\lambda_i(A+A')(t-s)}}{\lambda_i(A+A')}\right)\Bigg|_{t_0}^{t} \\ &= \lambda_i(P(t_0))\operatorname{tr}(e^{(A+A')(t-t_0)}) - \lambda_1(Q) \\ &\quad \cdot \sum_{i=1}^{n}\frac{1}{\lambda_i(A+A')} + \lambda_1(Q)\sum_{i=1}^{n}\frac{e^{\lambda_i(A+A')(t-t_0)}}{\lambda_i(A+A')} \\ &= \lambda_i(P(t_0))\operatorname{tr}(e^{(A+A')(t-t_0)}) - \lambda_1(Q) \\ &\quad \cdot \operatorname{tr}((A+A')^{-1}) \\ &\quad + \lambda_1(Q)\sum_{i=1}^{n}\frac{e^{\lambda_i(A+A')(t-t_0)}}{\lambda_i(A+A')}.\end{aligned}$$

This completes the proof. □

From the proof, we can easily modify the first term on the right-hand side of the inequality in Theorem 3.1 to obtain the following:

Theorem 3.2: Suppose that the real matrix A is stable and $A + A'$ is nonsingular, then we have

$$\begin{aligned}\operatorname{tr}(P(t)) &\le \operatorname{tr}(P(t_0))\lambda_1(e^{(A+A')(t-t_0)}) - \lambda_1(Q) \\ &\quad \cdot \operatorname{tr}((A+A')^{-1}) + \lambda_1(Q)\sum_{i=1}^{n}\frac{e^{\lambda_i(A+A')(t-t_0)}}{\lambda_i(A+A')}.\end{aligned}$$

Another direct approach is to use Lemma 2.1 to obtain the following result.

Theorem 3.3: We have following estimates for the trace of the solution of (1):

$$z_1(t) \le \operatorname{tr}(P(t)) \le z_2(t)$$

where $z_1(t)$ and $z_2(t)$ are the solutions of the following scalar differential equations:

$$\begin{aligned}\dot z_1(t) &= \lambda_n(A+A')z_1(t) + \operatorname{tr}(Q), \quad z_1(t_0) = \operatorname{tr}(P(t_0)) \\ \dot z_2(t) &= \lambda_1(A+A')z_2(t) + \operatorname{tr}(Q), \quad z_2(t_0) = \operatorname{tr}(P(t_0)).\end{aligned}$$

Proof: Taking the trace on both sides of (1), with $y(t) = \operatorname{tr}(P(t))$ we obtain

$$\dot y(t) = \operatorname{tr}((A+A')P(t)) + \operatorname{tr}(Q). \tag{3}$$

Applying Lemma 2.1, we have

$$\begin{aligned}\lambda_n(A+A')\operatorname{tr}(P(t)) &\le \operatorname{tr}((A+A')P(t)) \\ &\le \lambda_1(A+A')\operatorname{tr}(P(t)).\end{aligned}$$

Taking this into (3) and recalling that $\operatorname{tr}(P(t)) = y(t)$, we obtain

$$\begin{aligned}&\lambda_n(A+A')y(t) + \operatorname{tr}(Q) \\ &\quad \le \dot y(t) \le \lambda_1(A+A')y(t) + \operatorname{tr}(Q).\end{aligned}$$

Using the Gronwall–Bellman lemma, the desired result is obtained directly. □

If $\lambda_1(A + A') < 0$, we obtain an estimate for the trace of the solution of the algebraic Lyapunov matrix equation (2) and we have the following.

Corollary 3.1: If $\lambda_1(A + A') < 0$, then the trace of the solution of (2) has the following upper bound estimates:

$$\operatorname{tr}(P) \le -\lambda_1(Q)\operatorname{tr}((A+A')^{-1}) \tag{4}$$

$$\operatorname{tr}(P) \le \frac{\operatorname{tr}(Q)}{-\lambda_1(A+A')}. \tag{5}$$

Proof: In Theorem 3.1 or Theorem 3.2, letting t go to infinity, we can obtain (4). Similarly, (5) can be obtained from Theorem 3.3. □

Remarks:

1) Theorem 3.3 is the exact result obtained by Mori *et al.* [5], but the proof given here is much simpler than [5].
2) For the upper bound, it is hard to say whether Theorem 3.1, 3.2, or 3.3 is better. However, if we choose $Q = \alpha I$, our result improves Mori *et al.*'s result. To illustrate this, we prove that (4) is better than (5) for this case. In fact, if $\lambda_1(A + A') < 0$, then $\lambda_i(A + A') < 0$, and from (4), we have

$$\begin{aligned}\operatorname{tr}(P) &\le -\lambda_1(Q)\operatorname{tr}((A+A')^{-1}) \\ &= -\alpha\left(\frac{1}{\lambda_1(A+A')} + \cdots + \frac{1}{\lambda_n(A+A')}\right) \\ &\le \frac{-n\alpha}{\lambda_1(A+A')}\end{aligned}$$

from which we conclude that (4) is better than (5) for $Q = \alpha I$. From [8, (46)], we can also obtain (5). Therefore, our result also improves the bound obtained in [8].

In Lemma 2.1, the first set of inequalities is better than the second set of inequalities, and the latter is often used to obtain estimates for solutions of algebraic Lyapunov equations in the literature; therefore, we can expect that if the first set of inequalities in Lemma 2.1 is used, a better estimate for the trace of the solution of (2) can be obtained. This result is summarized in the next theorem.

Theorem 3.4: Suppose that $\mu_2(A) < 0$ and Q is a positive semidefinite matrix, then the solution P has the following upper bound estimate:

$$\operatorname{tr}(P) \le -\frac{\operatorname{tr}(Q)}{2\mu_2(A)} + \frac{\lambda_n(Q)(n\mu_2(A) - \operatorname{tr}(A))}{2\mu_2(A)\mu_2(-A)}.$$

If A is only stable, then we have the following lower bound estimate:

$$\operatorname{tr}(P) \ge \frac{\operatorname{tr}(Q)}{2\mu_2(-A)} + \frac{\lambda_n(Q)(\operatorname{tr}(A) + n\mu_2(-A))}{2\mu_2^2(-A)}.$$

Proof: From (2), we have

$$0 = 2\operatorname{tr}(\overline{A}P) + \operatorname{tr}(Q). \tag{6}$$

Applying Lemma 2.1 to the first term in (6), we obtain

$$0 \le 2\lambda_1(\overline{A})\,\mathrm{tr}\,(P) - 2\lambda_n(P)(n\lambda_1(\overline{A}) - \mathrm{tr}\,(A)) + \mathrm{tr}\,(Q)$$

i.e.,

$$-2\mu_2(A)\,\mathrm{tr}\,(P) \le -2\lambda_n(P)(n\mu_2(A) - \mathrm{tr}\,(A)) + \mathrm{tr}\,(Q).$$

Since $\mu_2(A) < 0$, from the above inequality, we have

$$\mathrm{tr}\,(P) \le \frac{\mathrm{tr}\,(Q)}{-2\mu_2(A)} + \frac{n\mu_2(A) - \mathrm{tr}\,(A)}{2\mu_2(A)}\lambda_n(P). \tag{7}$$

Notice that $n\mu_2(A) - \mathrm{tr}\,(A) = n\lambda_1(\overline{A}) - \mathrm{tr}\,(\overline{A}) = \Sigma_{i=1}^n \{\lambda_1(\overline{A} - \lambda_i(\overline{A})\} \ge 0$. We obtain

$$\frac{n\mu_2(A) - \mathrm{tr}\,(A)}{2\mu_2(A)} \le 0. \tag{8}$$

Let x be the eigenvector of P associated with the eigenvalue $\lambda_n(P)$. Then from (2), we can easily obtain

$$\lambda_n(P) = \frac{x'Qx}{2x'(-\overline{A})x} \ge \frac{\lambda_n(Q)}{2\lambda_1(-\overline{A})} = \frac{\lambda_n(Q)}{2\mu_2(-A)}.$$

Taking this into (7) with the aid of (8), we obtain the desired upper bound estimate.

Using a similar method, we can prove the lower bound estimate. This completes the proof. □

Remark: From the proof, we observe that the second term of the first inequality in Theorem 3.4 is nonpositive, so this bound is better than Mori *et al.*'s [5], Mrabti *et al.*'s [8] and [2, eq. (18)]. However, it is difficult to say whether Corollary 3.1 or Theorem 3.4 provides the better result. Because the second term of the second inequality in Theorem 3.4 is nonnegative, we have

$$\mathrm{tr}\,(P) \ge \frac{\mathrm{tr}\,(Q)}{2\mu_2(-A)}. \tag{9}$$

This is better than some known results in the current literature, for example, [8, eq. (47)]. There are some excellent lower bounds for the solution of algebraic Lyapunov equations (see [3] and [8]).

Under the assumption that $\mu_2(A) < 0$, i.e., $\lambda_1(A + A') < 0$, using majorization techniques [13], Komaroff [9] was able to obtain an excellent upper bound for the trace of the solution of (2). We state it in the following.

Theorem 3.5 [9]: Assume that $\lambda_1(A + A') < 0$, the trace of the solution of (2) has the following upper bound:

$$\mathrm{tr}\,(P) \le -\sum_{i=1}^{n} \frac{\lambda_i(Q)}{\lambda_i(A + A')}.$$

Remark: If $\lambda_1(A + A') < 0$, Theorem 3.5 is better than Corollary 3.1.

The assumption $\mu_2(A) < 0$ is commonplace for obtaining upper bounds (and some lower bounds) for the solution of the Lyapunov equation (see Theorem 3.5). However, for some stable matrices, the above assumption may be violated, hence the above upper bound for the trace of the solution becomes meaningless. For example, $A = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}$ is a stable matrix, however, $\lambda_1(A + A') = 0$. To overcome this difficulty, we want to make the following modification. From Lemma 2.2, we can see that for any stable matrix A, there exists a positive definite matrix F such that $\mu_F(A) < 0$. Let $T = \sqrt{F}$. From (2), we obtain

$$\begin{aligned}(T^{-1}PT^{-1})(TAT^{-1}) + (T^{-1}A'T)(T^{-1}PT^{-1})\\ + T^{-1}QT^{-1} = 0.\end{aligned} \tag{10}$$

Applying Lemma 2.1 and Theorem 3.4 to (10), we can easily obtain the following theorem.

Theorem 3.6: Select a positive definite matrix F satisfying $\mu_F(A) < 0$, then we have

$$\begin{aligned}\mathrm{tr}\,(P) \le{}& \frac{\lambda_1(F)\,\mathrm{tr}\,(F^{-1}Q)}{-2\mu_F(A)}\\ &+ \frac{\lambda_1(F)\lambda_n(F^{-1}Q)(n\mu_F(A) - \mathrm{tr}\,(A))}{2\mu_F(A)\mu_F(-A)}.\end{aligned}$$

Proof: Notice that $\mathrm{tr}\,(F^{-1}P) \ge \mathrm{tr}\,(P)/\lambda_1(F)$, then the proof follows easily. □

Corollary 3.2: Given a positive definite matrix F satisfying $\mu_F(A) < 0$, then

$$\mathrm{tr}\,(P) \le -\frac{\lambda_1(F)\,\mathrm{tr}\,(F^{-1}Q)}{\lambda_1(FAF^{-1} + A')}.$$

Applying Theorem 3.5 to (10), we can obtain the following generalization.

Theorem 3.7: Let F be a positive definite matrix satisfying $\mu_F(A) < 0$, then

$$\mathrm{tr}\,(P) \le -\lambda_1(F)\sum_{i=1}^{n} \frac{\lambda_i(F^{-1}Q)}{\lambda_i(FAF^{-1} + A')}.$$

□

Remark: When $F = I$ we can obtain Theorem 3.5. The advantage of Theorem 3.7 is that we do not use the assumption $\lambda_1(A + A') < 0$. The only assumption we have is the stability of A, which is a necessary and sufficient condition for the existence and uniqueness of a positive definite solution of the Lyapunov matrix equation.

Following the same idea, we can obtain an upper bound for the largest eigenvalue of the solution P.

Theorem 3.8: Let F be a positive definite matrix satisfying $\mu_F(A) < 0$, then

$$\lambda_1(P) \le \frac{\lambda_1(F)\lambda_1(F^{-1}Q)}{-2\mu_F(A)}.$$

In particular, if $\mu_2(A) < 0$, then we have

$$\lambda_1(P) \le \frac{\lambda_1(Q)}{-2\mu_2(A)}.$$

□

In Theorems 3.6 to 3.8, the matrix F is introduced to improve the upper bound estimates. The selection of F to obtain the tightest upper bound estimates is an open question. Lemma 2.2 shows that if A is stable, there exists a matrix F such that $\mu_F(A) < 0$. One procedure to find such an F is given in the proof of [11, Th. 4]. From (10), we observed the relationship between F and a similarity transformation on the system matrix A. This provides a way of optimizing the upper bounds given in this paper.

Next, we present an example to illustrate the generality of the results obtained in this paper.

Example: Let

$$A = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since $\mu_2(A) = 0.5\lambda_1(A + A') = 0$, we cannot use Theorem 3.4 or Theorem 3.5.

Choose $F = \begin{pmatrix} \varepsilon^2 & 0 \\ 0 & 1 \end{pmatrix}$, where $\varepsilon > 0$ is to be determined. We have

$$FAF^{-1} + A' = \begin{pmatrix} -2 & -2\varepsilon^2 \\ 2 & -2 \end{pmatrix}.$$

Thus, $\lambda_1(FAF^{-1}) = -2(1-\varepsilon)$ and $\lambda_2(FAF^{-1}) = -2(1+\varepsilon)$. Choose $\varepsilon = 0.5$. Applying Theorem 3.7, we have

$$\mathrm{tr}\,(P) \le -\left[\frac{1}{-2(1-\varepsilon)} + \frac{\varepsilon^{-2}}{-2(1+\varepsilon)}\right] = \frac{7}{3}.$$

Using Corollary 3.2, we have

$$\mathrm{tr}\,(P) \le \frac{1 + 1/\varepsilon^2}{-2(1-\varepsilon)} = 5.$$

From Theorem 3.6, we get

$$\mathrm{tr}\,(P) \le 5 - \tfrac{2}{3} = \tfrac{13}{3}.$$

From this example, Theorem 3.7 gives the best estimate. Applying Theorem 3.8, we obtain $\lambda_1(P) \le 4$. However, the results in [1], [2], [5], [8], and [9] cannot be used for this example.

It can be seen that using the modified Lyapunov equation (10), most of the known results in the current literature can be generalized, and better estimates can be obtained in this way. This is left to the readers.

IV. Conclusions

In this paper, new estimates for solutions of differential and algebraic Lyapunov matrix equations are obtained, generalizing some of the well-known results in the literature. Future research is directed to the application of this new approach to differential and algebraic Riccati equations.

References

[1] T. Mori and I. A. Derese, "A brief summary of the bounds on the solution of algebraic matrix equations in control theory," *Int. J. Contr.*, vol. 39, no. 2, pp. 247–256, 1984.

[2] N. Komaroff, "Upper bounds for the eigenvalues of the solution of the Lyapunov matrix equation," *IEEE Trans. Automat. Contr.*, vol. 35, no. 6, pp. 737–739, 1990.

[3] ——, "Simultaneous eigenvalue lower bounds for the Lyapunov matrix equation," *IEEE Trans. Automat. Contr.*, vol. 33, no. 1, pp. 126–128, 1988.

[4] B. H. Kwon, M. J. Youn, and Z. Bien, "On bounds of the Riccati and Lyapunov matrix equations," *IEEE Trans. Automat. Contr.*, vol. 30, no. 11, pp. 1134–1135, 1985.

[5] T. Mori, N. Fukuma, and M. Kuwahara, "Bounds in the Lyapunov matrix differential equation," *IEEE Trans. Automat. Contr.*, vol. 32, no. 1, pp. 55–57, 1987.

[6] Y. Fang, K. A. Loparo, and X. Feng, "Inequalities for the trace of matrix product," *IEEE Trans. Automat. Contr.*, vol. 39, no. 12, pp. 2489–2490, Dec. 1994.

[7] ——, "Robust stability and performance analysis: A new approach," in *Proc. IEEE Conf. Decision Contr.*, San Antonio, TX, Dec. 1993.

[8] M. Mrabti and A. Hmamed, "Bounds for the solution of the Lyapunov matrix equation—A unified approach," *Syst. Contr. Lett.*, vol. 18, pp. 73–81, 1992.

[9] N. Komaroff, "Upper summation and product bounds for solution eigenvalues of the Lyapunov matrix equation," *IEEE Trans. Automat. Contr.*, vol. 37, no. 7, pp. 1040–1042, 1992.

[10] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*. New York: Cambridge Univ. Press, 1991.

[11] Y. Fang, K. A. Loparo, and X. Feng, "A sufficient condition for the stability of a polytope of interval matrices," *Syst. Contr. Lett.*, vol. 23, no. 4, pp. 237–245, 1994.

[12] C. A. Desoer and M. Vidyasagar, *Feedback Systems: Input–Output Properties*. New York: Academic, 1975.

[13] A. W. Marshall and I. Olkin, *Inequalities: Theory of Majorization and Its Applications*. New York: Academic, 1979.

A Remark on the Stabilization of Partially Linear Composite Systems

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***Abstract*—In this paper, we study the global stabilization, by means of smooth state feedback, of partially linear composite systems. We show how to compute the stabilizing feedback thanks to a weak Lyapunov function for a nonlinear subsystem instead of a strict one.**

***Index Terms*—Feedback, global stabilization, Lyapunov function, nonlinear systems.**

I. Introduction

Many recent papers (see [1], [2], [6], and the references therein) addressed the problem of the global stabilization, by means of state feedback, of nonlinear control systems of the form

$$\begin{cases} \dot{x} = f(x, y) \\ \dot{y} = Ay + Bu \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, $u \in \mathbb{R}^k$, $A \in \mathcal{M}_{p,p}(\mathbb{R})$, $B \in \mathcal{M}_{p,k}(\mathbb{R})$, and f is a smooth vector field such that

h1) the pair (A, B) is stabilizable;

h2) the equilibrium $x = 0$ of $\dot{x} = f(x, 0)$ is globally asymptotically stable (G.A.S). In [6], the authors assumed that the dependence of $f(x, y)$ on y is of the following form;

h3) $f(x, y) = f(x, 0) + G(x, y)Cy$, with $C \in \mathcal{M}_{k,p}(\mathbb{R})$, and both C and B are of full rank.

They proved that there exist a matrix $K \in \mathcal{M}_{k,p}(\mathbb{R})$ and a symmetric positive definite matrix $P \in \mathcal{M}_{p,p}(\mathbb{R})$ satisfying the following three conditions:

H1) $P(A + BK) + (A + BK)^T P = -Q$, with Q symmetric positive (^{T} = transpose);

H2) $(Q^{1/2}, A + BK)$ detectable;

H3) $B^T P = C$, if and only if, the linear subsystem

$$\begin{cases} \dot{y} = Ay + Bu \\ \tilde{y} = Cy, \qquad \tilde{y} \in \mathbb{R}^k \end{cases} \tag{2}$$

is invertible, weakly minimum phase, and with CB symmetric positive definite.

Using these conditions, they showed that (1) is globally asymptotically stabilizable, and they gave the stabilizing feedback

$$u(x, y) = Ky - \tfrac{1}{2}G(x, y)^T \nabla V(x)$$

where V is a smooth Lyapunov function satisfying

$$\langle \nabla V, f(x, 0) \rangle < 0 \quad \forall x \in \mathbb{R}^n, \ x \neq 0. \tag{3}$$

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