Correspondence

A New General Sufficient Condition for Almost Sure Stability of Jump Linear Systems

Yuguang Fang

***Abstract*—In this paper, we study the almost sure stability of discrete-time jump linear systems with a finite-state Markov-form process. A general condition for almost sure stability, which is a necessary and sufficient condition for (scalar) one-dimensional systems, is derived. Many simpler testable sufficient conditions for almost sure stability are derived from this sufficient condition.**

***Index Terms*—Almost sure stability, jump linear systems, stochastic stability.**

I. Introduction

Many practical systems which are subject to abrupt changes, such as component and/or interconnection failure or random communication delays in automobile vehicles, can be modeled by jump linear stochastic systems in the form

$$x_{t+1} = H(\sigma_t)x_t + G(\sigma_t)u_t, \quad t \in Z^+ = \{0, 1, \cdots\} \tag{1}$$

where $\{\sigma_t\}$ is a finite-state Markov chain [5], [17], [18]. Therefore, a significant effort has been devoted to the optimal control of jump linear systems with a quadratic cost functional and to developing notions of controllability and observability for this class of systems. Many important results concerning the analysis and design of such systems have been obtained; see, for example, Ji and Chizeck [13], Mariton [18], and the references cited therein.

The stability of a dynamical system is one of the primary concerns in the design and synthesis of a control system. The study of stability of jump linear systems has attracted considerable attention. The earliest work can be traced back to Rosenbloom [19]. Bellman [1] and Bergen [2] studied the moment stability properties. Later, Bhuracha [4] used the idea developed in [1] to generalize Bergen's results and studied both the asymptotic stability and the exponential stability of the mean. Darkhovskii and Leibovich [7] investigated the second moment stability of systems where $\{\sigma_t\}$ is a semi-Markovian process and obtained necessary and sufficient conditions for second moment stability in terms of the Kronecker matrix product. This is an extension of Bhuracha's result. Kats and Krasovskii [16] and Bertram and Sarachik [3] used a stochastic version of Lyapunov's second method to study almost sure and moment stability. Recently, Ji *et al*. [15] and Feng *et al*. [11] used Lyapunov's second method, Costa and Fragoso [6] applied the Kronecker operators, to study the stability of (1) with a finite-state Markov chain form process and obtained necessary and sufficient conditions for the second moment stability. Costa and Fragoso [6] also obtained an interesting condition for almost sure stability. Fang *et al.* [8] and [9] and Fang [10] systematically studied both almost sure and δ-moment stability for (1) and obtained many sufficient conditions for both kinds of stability. The relationship between almost sure stability and δ-moment stability was characterized using the large deviation theory. Inspired by this relationship, we derive some less restrictive conditions for almost sure stability.

This paper is the continuing research for the stochastic stability of jump linear systems. We present a new sufficient condition for almost sure stability for jump linear systems with a finite-state Markov chain process. From this general condition, we give a few simple testable conditions for almost sure stability.

Manuscript received February 21, 1996; revised April 14, 1996.

The author is with the Department of Electrical and Computer Engineering, College of Engineering, Boston University, Boston, MA 02215 USA (e-mail: yxf2@engc.bu.edu).

Publisher Item Identifier S 0018-9286(97)02036-9.

II. Almost Sure Stability

We concentrate on the discrete-time jump linear system given by

$$x_{k+1} = H(\sigma_k)x_k, \quad k \geq 0 \tag{2}$$

where σ_k is either an finite-state independently identically distributed (i.i.d.) process in the state space $\underline{N} = \{1, 2, \cdots, N\}$ with probability distribution $P\{\sigma_0 = j\} = p_j$ for $j \in \underline{N}$ or a finite-state and time-homogeneous Markov chain with state space $\underline{N}$, transition probability matrix $P = (p_{ij})_{N \times N}$, and initial distribution $p = (p_1, \cdots, p_N)$. Let $\pi = (\pi_1, \cdots, \pi_N)$ be the unique invariant probability distribution of $\{\sigma_k\}$ if it is an ergodic Markov chain. We will use the following notation in this paper. Let A^T denote the matrix transpose. For symmetric matrices A and B, $A < B$ denotes that $B - A$ is pd. For any matrices X and Y, $X <_e Y$ denotes the elementwise inequalities. Let $\|x\|_2$ denote the Euclidean norm and $\|A\|_2$ denote the induced matrix norm, which can be given by $\sqrt{\lambda_{\max}(A^TA)}$, $\lambda_{\max}(\cdot)$, and $\lambda_{\min}(\cdot)$ denote the largest and smallest eigenvalues, respectively. $\rho(\cdot)$ denotes the spectral radius.

Definition 2.1: The jump linear system (2) with a Markovian form process $\{\sigma_k\}$ as specified above is said to be *almost surely (asymptotically) stable*, if for any $x_0 \in R^n$ and any initial distribution p of σ_k

$$P\Big\{\lim_{k\to\infty} \|x_k(x_0, \omega)\| = 0\Big\} = 1.$$

The following is one of our main results.

Theorem 2.1: Suppose that $\{\sigma_k\}$ is a finite-state Markov chain with transition matrix $P = (p_{ij})$. If there exist positive definite (pd) matrices $P(1), P(2), \cdots, P(N)$ such that

$$\sup_{\|x\|_2=1} \prod_{j=1}^{N} \left(\frac{x^TH^T(i)P(j)H(i)x}{x^TP(i)x}\right)^{p_{ij}} < 1, \quad \forall i \in \underline{N} \tag{3}$$

then (2) is almost surely stable.

Proof: Define the Lyapunov function

$$V(x_k, \sigma_k) = \left(x_k^TP(\sigma_k)x_k\right)^{\delta/2}.$$

Then, we have

$$\begin{aligned}
&\Delta V(x, i)\\
&\quad = E\big\{(x_{k+1}^TP(\sigma_{k+1})x_{k+1})^{\delta/2} \,|\, x_k = x, \sigma_k = i\big\}\\
&\qquad - (x^TP(i)x)^{\delta/2}\\
&\quad = E\big\{(x^TH^T(\sigma_k)P(\sigma_{k+1})H(\sigma)x)^{\delta/2} \,|\, \sigma_k = i\big\}\\
&\qquad - (x^TP(i)x)^{\delta/2}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{N} p_{ij}(x^T H^T(i)P(j)H(i)x)^{\delta/2} - (x^T P(i)x)^{\delta/2} \\
&= (x^T P(i)x)^{\delta/2} \\
&\quad \times \left[\sum_{j=1}^{N} p_{ij}\left(\frac{x^T H^T(i)P(j)H(i)x}{x^T P(i)x}\right)^{\delta/2} - 1\right] \qquad (4)\\
&= (x^T P(i)x)^{\delta/2} \\
&\quad \times \left\{\left[\left(\sum_{j=1}^{N} p_{ij}\left(\frac{x^T H^T(i)P(j)H(i)x}{x^T P(i)x}\right)^{\delta/2}\right)^{2/\delta}\right]^{\delta/2} - 1\right\}.
\end{aligned}
$$

As $\sum_{j=1}^{N} p_{ij} = 1$, we have from L'Hospital's rule

$$
\begin{aligned}
&\lim_{\delta\to 0}\left[\sum_{j=1}^{N} p_{ij}\left(\frac{x^T H^T(i)P(j)H(i)x}{x^T P(i)x}\right)^{\delta/2}\right]^{2/\delta} \\
&\quad = \prod_{j=1}^{N}\left(\frac{x^T H^T(i)P(j)H(i)x}{x^T P(i)x}\right)^{p_{ij}}. \qquad (5)
\end{aligned}
$$

Suppose that (3) holds; there exists ρ, $0 < \rho < 1$, such that

$$
\sup_{\|x\|=1} \prod_{j=1}^{N}\left(\frac{x^T H^T(i)P(j)H(i)x}{x^T P(i)x}\right)^{p_{ij}} < \rho < 1. \qquad (6)
$$

Then, we can obtain that there exists a $\delta > 0$ such that for any $x \in R^n$ satisfying $\|x\| = 1$ and for any $i \in \underline{N}$, the following holds:

$$
\left[\sum_{j=1}^{N} p_{ij}\left(\frac{x^T H^T(i)P(j)H(i)x}{x^T P(i)x}\right)^{\delta/2}\right]^{2/\delta} < \rho. \qquad (7)
$$

In fact, suppose that this is not true, then for any $\delta > 0$, there exists $i \in \underline{N}$ and x satisfying $\|x\| = 1$ such that

$$
\left[\sum_{j=1}^{N} p_{ij}\left(\frac{x^T H^T(i)P(j)H(i)x}{x^T P(i)x}\right)^{\delta/2}\right]^{2/\delta} \geq \rho.
$$

As $\underline{N}$ is a finite set and $S^n \triangleq \{x \| \|x\| = 1\}$ is compact, without loss of generality, we can choose $i \in \underline{N}$ and a convergent sequence δ_k satisfying $\delta_k \downarrow 0+$ and a convergent sequence x_k satisfying $\lim_{k\to\infty} x_k = x_0$ and $\|x_0\| = 1$ such that

$$
\left[\sum_{j=1}^{N} p_{ij}\left(\frac{x_k^T H^T(i)P(j)H(i)x_k}{x_k^T P(i)x_k}\right)^{\delta_k/2}\right]^{2/\delta_k} \geq \rho. \qquad (8)
$$

For notational simplicity, let

$$
M_{ij}(x) = \frac{x^T H^T(i)P(j)H(i)x}{x^T P(i)x}.
$$

Since $M_{ij}(x)$ is continuous on S^n, for any $\varepsilon > 0$, there exists $K > 0$ such that whenever $k > K$, we have $M_{ij}(x_k) \leq M_{ij}(x_0) + \varepsilon$. From (8), we obtain

$$
\left(\sum_{j=1}^{N} p_{ij}(M_{ij}(x_0) + \varepsilon)^{\delta_k/2}\right)^{2/\delta_k} \geq \rho.
$$

From this and (5), we have

$$
\prod_{j=1}^{N}(M_{ij} + \varepsilon)^{p_{ij}} \geq \rho.
$$

Letting ε go to zero, we obtain

$$
\prod_{j=1}^{N} M_{ij}^{p_{ij}}(x_0) \geq \rho.
$$

This contradicts (6), thus the claim in (7) is proved.

Taking (7) into (4), we obtain

$$
\begin{aligned}
\Delta V(x,i) &\leq (x^T P(i)x)^{\delta/2}(\rho^{\delta/2} - 1) \\
&= -(1 - \rho^{\delta/2})(x^T P(i)x)^{\delta/2} < 0
\end{aligned}
$$

for any $x \neq 0$. From the stochastic version of Lyapunov's second method, we conclude that (2) is δ-moment stable, hence is almost surely stable. This completes the proof. □

From this theorem, we can obtain the following criterion.

Corollary 2.2: Suppose that $\{\sigma_k\}$ is a finite-state Markov chain with probability transition matrix $P = (p_{ij})$. If there exist pd matrices $P(1), P(2), \cdots, P(N)$ such that

$$
\prod_{j=1}^{N} \lambda_{\max}[H^T(i)P(j)H(i)P^{-1}(i)]^{p_{ij}} < 1, \quad (i = 1, 2, \cdots, N)
$$

then (2) is almost surely stable.

Proof: Using the following fact, $\max_{x\neq 0} \frac{x^T Q x}{x^T P x} = \lambda_{\max}(P^{-1/2} Q P^{-1/2}) = \lambda_{\max}(QP^{-1})$ and maximizing each term in the product of (2), we can obtain the proof. □

Next, we want to show that Theorem 2.1 provides a very general sufficient condition for almost sure stability of (2). For a one-dimensional system, the sufficient condition in Theorem 2.1 is also necessary. And if (2) is second moment stable, then (3) is also necessary.

Corollary 2.3: Suppose that (2) is a one-dimensional system with $|H(i)| \triangleq a_i \neq 0$ $(i = 1, 2, \cdots, N)$ and that $\{\sigma_k\}$ is a finite-state irreducible Markov chain with ergodic measure π, then a necessary and sufficient condition for (2) to be almost surely stable is that there exist N positive numbers $P(1), P(2), \cdots, P(N)$ such that (3) holds.

Proof: We only need to prove the necessity. We first prove that $\mathrm{Im}(P - I) = \{z \in \mathbf{R}^N : \pi z = 0\}$, where $\mathrm{Im}(A)$ denotes the image of linear mapping A. In fact, $\forall u \in \mathrm{Im}(P - I)$, there exists a $v \in \mathbf{R}^N$ such that $u = (P - I)v$, which implies, together with the identity $\pi P = \pi$, that $\pi u = \pi(P - I)v = (\pi P - \pi)v = 0$; hence $u \in \{z : \pi z = 0\}$, and this implies that $\mathrm{Im}(P - I) \subseteq \{z : \pi z = 0\}$. However, π is the unique solution to the equation

$$
\begin{cases}
\pi(P - I) = 0 \\
\pi(1, \cdots, 1)^T = 1.
\end{cases}
$$

We obtain that rank $(P - I) = N - 1$, so $\dim(\mathrm{Im}(P - I)) = N - 1$. Moreover, $\dim(\{z : \pi z = 0\}) = N - 1$, and we conclude that $\mathrm{Im}(P - I) = \{z : \pi z = 0\}$. Now let us choose

$$
z = -a + \frac{\pi a}{\pi \pi^T}\pi^T \in \mathbf{R}^N
$$

where $a = (\log a_1, \log a_2, \cdots, \log a_N)^T$. Since $\pi z = -\pi a + \pi a = 0$, i.e., $z \in \{z : \pi z = 0\} = \mathrm{Im}(P - I)$, there exists a $y \in \mathbf{R}^N$ such that $z = (P - I)y$, i.e.,

$$
(P - I)y + a = \frac{\pi a}{\pi \pi^T}\pi^T. \qquad (9)
$$

Suppose that (2) is almost surely stable. From [8] we have $a_1^{\pi_1} a_2^{\pi_2} \cdots a_N^{\pi_N} < 1$, i.e.,

$$
\pi_1 \log a_1 + \pi_2 \log a_2 + \cdots + \pi_N \log a_N < 0, \text{ or } \pi a < 0. \qquad (10)
$$

Now choose $P(i) = e^{2y_i} > 0$ $(i \in \underline{N})$; from (9) and (10) we have

$$
(P - I)\begin{pmatrix} \log\sqrt{P(1)} \\ \log\sqrt{P(2)} \\ \vdots \\ \log\sqrt{P(N)} \end{pmatrix} + \begin{pmatrix} \log a_1 \\ \log a_2 \\ \vdots \\ \log a_N \end{pmatrix} < 0
$$

which is equivalent to

$$\sum_{j=1}^{N} p_{ij} \log \sqrt{\frac{P(j)}{P(i)}} + \sum_{j=1}^{N} p_{ij} \log a_i < 0, \quad \forall i \in \underline{N}.$$

Thus, we finally arrive at

$$\prod_{j=1}^{N} \left(\frac{P(j)}{P(i)} a_i^2 \right)^{p_{ij}} < 1, \quad \forall i \in \underline{N}.$$

This completes the proof of the necessity. □

Corollary 2.4: If (2) is second moment stable, then there exist pd matrices $P(1), \cdots, P(N)$ such that (3) holds.

Proof: Suppose that (2) is second moment stable, then from [13], there exist pd matrices $P(1), P(2), \cdots, P(N)$ such that

$$\sum_{j=1}^{N} p_{ij} H^T(i) P(j) H(i) - P(i) = -I, \quad i = 1, 2, \cdots, N.$$

For any $x \neq 0$, we have

$$\sum_{j=1}^{N} p_{ij} x^T H^T(i) P(j) H(i) x = x^T P(i) x - x^T x$$

or

$$\sum_{j=1}^{N} p_{ij} \frac{x^T H^T(i) P(j) H(i) x}{x^T P(i) x} = 1 - \frac{x^T x}{x^T P(i) x}.$$

Using the inequality

$$\beta_1^{\alpha_1} \beta_2^{\alpha_2} \cdots \beta_N^{\alpha_N} \leq \alpha_1 \beta_1 + \alpha_2 \beta_2 + \cdots + \alpha_N \beta_N, \quad \left(\alpha_i \geq 0, \sum_{i=1}^{N} \alpha_i = 1 \right)$$

we obtain

$$\prod_{j=1}^{N} \left(\frac{x^T H^T(i) P(j) H(i) x}{x^T P(i) x} \right)^{p_{ij}} \leq 1 - \frac{x^T x}{x^T P(i) x} \leq 1 - \lambda_{\min}(P^{-1}(i)) < 1.$$

From this, we can conclude that (3) holds. □

In the derivation of the proof of Theorem 2.1, we have used the Lyapunov function: $V(x_k, \sigma_k) = (x_k^T P(\sigma_k) x)^{\delta/2}$. It is easy to see that x_k is measurable with respect to the σ-algebra generated by $\sigma_0, \sigma_1, \cdots, \sigma_{k-1}$. It is reasonable to construct the Lyapunov function, $V(x_k, \sigma_{k-1}) = (x_k^T P(\sigma_{k-1}) x_k)^{\delta/2}$, as we observed before. Then we can obtain the following:

$$\begin{aligned} \Delta V(x, i) &= E\{V(x_{k+1}, \sigma_k) \mid x_k = x, \sigma_{k-1} = i\} - V(x, i) \\ &\times E\left\{ \left[x_k^T H^T(\sigma_k) P(\sigma_k) H(\sigma_k) x_k \right]^{\delta/2} - (x^T P(i) x)^{\delta/2} \right\} \\ &= \sum_{j=1}^{N} p_{ij} [x^T H^T(j) P(j) H(j) x]^{\delta/2} - (x^T P(i) x)^{\delta/2}. \end{aligned}$$

From this consideration, the following results can be achieved.

Theorem 2.5: If there exist pd matrices $P(1), P(2), \cdots, P(N)$ such that

$$\max_{\|x\|_2 = 1} \prod_{j=1}^{N} \left(\frac{x^T H^T(j) P(j) H(j) x}{x^T P(i) x} \right)^{p_{ij}} < 1, \quad \forall i \in \underline{N} \tag{11}$$

then (2) is almost surely stable.

An easier testable condition is given by the following corollary.

Corollary 2.6: If there exist pd matrices $P(1), P(2), \cdots, P(N)$ such that

$$\prod_{j=1}^{N} (\lambda_{\max}(H^T(j) P(j) H(j) P^{-1}(i)))^{p_{ij}} < 1, \quad \forall i \in \underline{N} \tag{12}$$

then (2) is almost surely stable.

Corollary 2.7: If there exists a nonsingular matrix M such that

$$P\big(\log \|MH(1)M^{-1}\|_2, \cdots, \log \|MH(N)M^{-1}\|_2 \big)^T <_e 0 \tag{13}$$

then the system (2) is almost surely stable.

Proof: Notice that for any nonsingular matrix M, the matrix norm $\|A\|$ induced by $\|Mx\|$ is given by $\|MAM^{-1}\|$. In (12), choosing $P(1) = P(2) = \cdots = P(N) = M^T M$, we have

$$\begin{aligned} &\lambda_{\max}(H^T(j) P(j) H(j) P(i)^{-1}) \\ &= \lambda_{\max}(H^T(j) M^T M H(j) M^{-1} [M^{-1}]^T) \\ &= \lambda_{\max}([MH(j)M^{-1}]^T [MH(j)M^{-1}]) \\ &= \|MH(j)M^{-1}\|_2^2. \end{aligned}$$

Taking this into (12) and taking the logarithm on both sides, we obtain

$$\sum_{j=1}^{N} p_{ij} \log \|MH(j)M^{-1}\|_2 < 0, \quad i = 1, 2, \cdots, N.$$

From this, we can prove the corollary. □

For more special cases of $H(i)$, we can obtain simpler criteria. We present one in the following.

Corollary 2.8: Suppose that $H(i)$ $(i = 1, 2, \cdots, N)$ are all lower triangular forms with spectral radii $\rho(H(i))$. Then (2) is almost surely stable if

$$P(\log \rho(H(1)), \cdots, \log \rho(H(N)))^T <_e 0. \tag{14}$$

In particular, if $H(1), H(2), \cdots, H(N)$ pairwise commute, then (2) is almost surely stable if (14) holds.

Proof: For lower triangular matrices $H(i)$, in Corollary 2.7 choosing M to be a diagonal matrix with diagonal elements $1, \epsilon, \cdots, \epsilon^{n-1}$, we complete the proof. □

Remark: When the form process $\{\sigma_k\}$ is a finite-state i.i.d. process or a finite-state ergodic Markov chain, Fang *et al.* [8] obtained a similar result.

As we dealt with the δ-moment stability in [8], we can transform the high-dimensional system into a one-dimensional system. Then applying our result for one-dimensional system, we can obtain the following.

Corollary 2.9: If there exists a matrix norm $\|\cdot\|$ satisfying the multiplicative property (i.e., $\|AB\| \leq \|A\|\|B\|$) such that

$$P(\log \|H(1)\|, \cdots, \log \|H(N)\|)^T + (P - I)y <_e 0 \tag{15}$$

has a solution y, then (2) is almost surely stable. In particular, (2) is almost surely stable if

$$P(\log \|H(1)\|, \cdots, \log \|H(N)\|)^T <_e 0. \tag{16}$$

Proof: Notice that the almost sure stability of (2) is implied by the almost sure stability of the system $z_{k+1} = \|H(\sigma_k)\| z_k$. Suppose that (15) has a solution y, letting $P(i) = e^{2y_i}$, we can easily verify that (12) holds with $H(j)$ replaced by $\|H(j)\|$. From Corollary 2.6, the system $z_{k+1} = \|H(j)\| z_k$ is almost surely stable. This completes the proof of the first part. The second part is the special case of the first part, $y = 0$. □

In [8], we have proved the following result for δ-moment stability.

Theorem 2.10: Let $D = \text{diag}\{\|H(1)\|^\delta, \|H(2)\|^\delta, \cdots, \|H(N)\|^\delta\}$, then for $\delta > 0$, (2) is δ-moment stable if there exists a matrix norm $\|\cdot\|$ such that $\rho(PD) < 1$.

This theorem can be used to give a sufficient condition for almost sure stability.

Corollary 2.11: Let $D = \text{diag}\{\|H(1)\|^\delta, \|H(2)\|^\delta, \cdots, \|H(N)\|^\delta\}$, then (2) is almost surely stable if there exists a matrix norm $\|\cdot\|$ and a $\delta > 0$ such that $\rho(PD) < 1$.

There is a subtle relationship between Corollary 2.9 and Theorem 2.10. We use a special sufficient condition—the second part of Corollary 2.9—to illustrate this relationship. We show that if (16) holds, then $\rho(PD) < 1$. In fact, from (16), we have

$$\prod_{j=1}^{N} \|H(j)\|^{p_{ij}} < 1, \quad i = 1, 2, \cdots, N. \tag{17}$$

For any $i \in \{1, 2, \cdots, N\}$, we have

$$\lim_{\delta \to 0} \left(\sum_{j=1}^{N} p_{ij} \|H(j)\|^\delta \right)^{1/\delta} = \prod_{j=1}^{n} \|H(j)\|^{p_{ij}}.$$

From this and (17), there exists a $\delta > 0$ such that

$$\left(\sum_{j=1}^{N} p_{ij} \|H(j)\|^\delta \right)^{1/\delta} < 1.$$

From this, we can easily deduce that

$$PD(1, 1, \cdots, 1)^T <_e (1, 1, \cdots, 1)^T.$$

From [8, Lemma 3.1], we have $\rho(PD) < 1$. This means that (16) is a stronger condition for almost sure stability. Obviously, (16) is easier to test.

Since an i.i.d. process is also a finite Markov chain, all sufficient conditions developed above are almost valid for the systems with an i.i.d. finite-state process, and all sufficient conditions are much simpler. For example, [9, Th. 3.5] can be obtained from Theorem 2.5.

Notice that in deriving all above sufficient conditions, we do not assume that the form process $\{\sigma_k\}$ is ergodic. If the form process is ergodic, simpler conditions can be obtained for almost sure stability. The form process with the ergodic measure is just like a finite-state i.i.d. process for the purpose of almost sure stability. (However, this is not true for moment stability as we showed in [8, Example 3.1].) We can easily obtain the following useful result.

Corollary 2.12: Suppose that the form process $\{\sigma_k\}$ is a finite-state ergodic Markov process with ergodic measure $\pi = \{\pi_1, \pi_2, \cdots, \pi_N\}$. Then (2) is almost surely stable if there exists a matrix norm $\|\cdot\|$ such that

$$\|H(1)\|^{\pi_1} \|H(2)\|^{\pi_2} \cdots \|H(N)\|^{\pi_N} < 1. \tag{18}$$

Proof: Using the matrix norm to reduce (2) to a one-dimensional system, then applying the necessary and sufficient condition for the almost sure stability of one-dimensional systems, we can complete the proof (see the similar proof in Corollary 2.9). □

Remark: This result can also be obtained from [6, Th. 4].

III. Illustrative Examples

In this section, we present a few examples to show how to use the criteria developed in this paper.

Example 3.1 [8]: Let $H(1) = 1.9$, $H(2) = 0.5$, and $P = \begin{pmatrix} 0.1 & 0.9 \\ 0.8 & 0.2 \end{pmatrix}$. In [8], we showed that (2) is first-moment stable from Theorem 2.10, hence it is almost surely stable. Here, we want to apply Corollary 2.9. If we let $y = (-0.3, 0.3)^T$, we have

$$\begin{pmatrix} 0.1 & 0.9 \\ 0.8 & 0.2 \end{pmatrix} \begin{pmatrix} \log 1.9 \\ \log 0.5 \end{pmatrix} + \left[\begin{pmatrix} 0.1 & 0.9 \\ 0.8 & 0.2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} -0.3 \\ 0.3 \end{pmatrix} = \begin{pmatrix} -0.0196 \\ -0.1051 \end{pmatrix} <_e \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

From Corollary 2.9, (2) is almost surely stable. The general solution for linear inequalities can be found by the successive approximation algorithm. Since the process $\{\sigma_k\}$ is ergodic with ergodic measure $\pi = (8/17, 9/17)$, Corollary 2.12 can also apply to this case. In fact, $1.9^{8/17} 0.5^{9/17} = 0.9372 < 1$.

Example 3.2: Let

$$H(1) = \begin{pmatrix} 0.5 & 1 \\ 0 & 0.5 \end{pmatrix}, \quad H(2) = \begin{pmatrix} 1 & 0 \\ 0.1 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 0.4 & 0.6 \\ 0.8 & 0.2 \end{pmatrix}.$$

In Corollary 2.7, choose $M = \text{diag}\{1, 5\}$. We have

$$\|MH(1)M^{-1}\|_2^{0.4} \|MH(2)M^{-1}\|_2^{0.6} = 0.9519$$
$$\|MH(1)M^{-1}\|_2^{0.8} \|MH(2)M^{-1}\|_2^{0.2} = 0.7075.$$

From the above, we know that (13) is satisfied, so (2) is almost surely stable.

Example 3.3: Let

$$H(1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, H(2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, P = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}.$$

In Theorem 2.5, choose $P(1) = P(N) = I$. We can easily obtain that

$$\begin{aligned} \max_{\|x\|_2=1} \prod_{j=1}^{2} \left(\frac{x^T H^T(j) P(j) H(j) x}{x^T P(i) x} \right)^{p_{ij}} &= \max_{\|x\|_2=1} \|H(1)x\|_2 \|H(2)x\|_2 \\ &= \max_{0 \le \theta \le 2\pi} |\sin\theta \cos\theta| = \frac{1}{2} < 1. \end{aligned}$$

From Theorem 2.5, the system is almost surely stable. However, if we choose the Euclidean norm in Theorem 2.10, then $\|H(1)\| = \|H(2)\|_2 = 1$, then for any $\delta > 0$, $\rho(PD) = \rho(P) = 1$; hence we cannot apply Theorem 2.10. The same problems happen for some of the other sufficient conditions.

IV. Conclusion

This paper address the almost sure stability problems for a class of linear stochastic systems called jump linear systems. A new general sufficient condition for almost sure stability is obtained. From this condition, many simpler testable conditions are derived. However, this condition involves a minimax problem, which is still difficult to solve. The numerical consideration for this problem will be investigated in the future.

Acknowledgment

The author would like to express his gratitude to the associate editor and the anonymous referees for their constructive comments for the improvement of this paper. Thanks are also due to K. Loparo and X. Feng at Case Western Reserve University for their inspiring discussions.

REFERENCES

[1] R. Bellman, "Limiting theorems for noncommutative operators," *Duke Math. J.*, vol. 21, p. 456, 1960.

[2] A. R. Bergen, "Stability of systems with randomly time-varying parameters," *IRE Trans.*, vol. AC-5, p. 265, 1960.

[3] J. E. Bertram and P. E. Sarachik, "Stability of circuits with randomly time-varying parameters," *IRE Trans.*, Special Issue, vol. PGIT-5, p. 260, 1959.

[4] B. H. Bharucha, "On the stability of randomly varying systems," Ph.D. dissertation, Dept. Elec. Eng., Univ. Calif., Berkeley, July 1961.

[5] H. J. Chizeck, A. S. Willsky, and D. Castanon, "Discrete-time Markovian-jump quadratic optimal control," *Int. J. Contr.*, vol. 43, no. 1, pp. 213–231, 1986.

[6] O. L. V. Costa and M. D. Fragoso, "Stability analysis for discrete-time linear systems with Markovian jumping parameters," *J. Math. Anal. Appl.*, vol. 179, pp. 154–178, 1993.

[7] B. S. Darkhovskii and V. S. Leibovich, "Statistical stability and output signal moments of a class of systems with random variation of structure," *Auto. Remote Contr.*, vol. 32, no. 10, pp. 1560–1567, 1971.

[8] Y. Fang, K. A. Loparo, and X. Feng, "Almost sure and δ-moment stability of jump linear systems," *Int. J. Contr.*, vol. 59, no. 5, pp. 1281–1307, 1994.

[9] ——, "Stability of discrete time jump linear systems," *J. Math. Syst., Estimation Contr.*, vol. 5, no. 3, pp. 275–321, 1995.

[10] Y. Fang, "Stability analysis of linear control systems with uncertain parameters," Ph.D. dissertation, Dept. Systems, Control, and Industrial Engineering, Case Western Reserve Univ., Jan. 1994.

[11] X. Feng, K. A. Loparo, Y. Ji, and H. J. Chizeck, "Stochastic stability properties of jump linear systems," *IEEE Trans. Automat. Contr.*, vol. 37, no. 1, pp. 38–53, 1992.

[12] R. A. Horn and C. R. Johnson, *Matrix Analysis*. Cambridge, MA: Cambridge Univ. Press, 1985.

[13] Y. Ji and H. J. Chizeck, "Jump linear quadratic Gaussian control: Steady state solution and testable conditions," *Contr. Theory Advanced Tech.*, vol. 6, no. 3, pp. 289–319, 1990.

[14] ——, "Controllability, stabilizability, and continuous time Markovian jump linear quadratic control," *IEEE Trans. Automat. Contr.*, vol. 35, no. 7, pp. 777–788, 1990.

[15] Y. Ji, H. J. Chizeck, X. Feng, and K. A. Loparo, "Stability and control of discrete-time jump linear systems," *Contr. Theory Advanced Tech.*, vol. 7, no. 2, pp. 247–270, 1991.

[16] I. I. Kats and N. N. Krasovskii, "On the stability of systems with random parameters," *Prkil. Met. Mek.*, vol. 24, p. 809, 1960.

[17] R. Krtolica, Ü. Özgüner, H. Chan, H. Göktas, J. Winkelman, and M. Liubakka, "Stability of linear feedback systems with random communication delays," in *1991 Amer. Contr. Conf. Proc.*, Boston, MA, June 26–28, 1991, pp. 2648–2653.

[18] M. Mariton, *Jump Linear Systems in Automatic Control*. New York: Marcel Dekker, 1990.

[19] A. Rosenbloom, "Analysis of linear systems with randomly time-varying parameters," in *Proc. Symp. Inf. Nets*, Poly. Inst. Brooklyn, 1954, vol. III, p. 145.

Calculation of the Minimal Dimension kth-order Robust Servoregulator

Wei Kang and Jie Huang

***Abstract*— The design of a minimal dimension kth-order robust servoregulator requires calculation of the minimal polynomial of a class of matrices defined by the given exosystem. A characterization of the minimal polynomial of this class of matrices was given recently in [8] when the given exosystem is semisimple. This paper will further provide a characterization of this class of matrices in the general case. This result leads to a straightforward and efficient procedure to calculate the minimal dimension kth-order robust servoregulator.**

***Index Terms*—Linear algebra, nonlinear control, robust nonlinear servomechanism.**

I. INTRODUCTION

Feedback design for the servomechanism problem can be challenging when a system is highly nonlinear. Methods of finding the kth-order approximate solution of the problem for nonlinear systems with or without uncertainty are introduced in [9] and [6]. The notion of kth-order robust control was proposed in [6] to approximately solve the robust nonlinear servomechanism problem (or robust output regulation). Later it is shown that the kth-order robust control can actually solve the robust nonlinear servomechanism problem under additional assumptions [7], [3].

It was shown that the kth-order robust nonlinear servomechanism problem can be solved by a linear controller that contains an internal model of a dynamic system called k-fold exosystem, a nonlinear analog of the well-known *internal model principle* of linear regulation theory. Therefore, the design of a minimal dimension kth-order robust controller requires the calculation of the minimal polynomial of the k-fold exosystem. The k-fold exosystem is a linear autonomous system generated by the linearized exosystem. A characterization of the minimal polynomial of the k-fold exosystem was given recently in [8] where the given exosystem is semisimple. This paper will further provide a characterization of the k-fold exosystem in the general case. This result leads to a straightforward and efficient procedure to calculate the minimal dimension kth-order robust servoregulator.

The matrix in the k-fold exosystem is also the matrix of a linear differential operator, which arises in some related control problems such as nonlinear optimal control, nonlinear H_∞ control, and feedback linearization. The result in this paper is believed to be helpful to understand the nature of the approximate solutions to these problems.

The rest of this paper is organized as follows. Section II summarizes the results on the nonlinear robust servomechanism theory following the lines of [6]. Section III gives a characterization of the minimal polynomial of the k-fold exosystem. In Section IV, we close this paper with some remarks.

Manuscript received March 6, 1995; revised June 13, 1996. The work of the second author was supported in part by the Hong Kong Research Grants Council under Grant CUHK 380/96E.

W. Kang is with the Mathematics Department, Naval Postgraduate School, Monterey, CA 93943 USA.

J. Huang is with the Department of Mechanical and Automation Engineering, The Chinese University of Hong Kong, Shatin, New Territories, Hong Kong.

Publisher Item Identifier S 0018-9286(97)02041-2.