# Inequalities for the Trace of Matrix Product

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***Abstract*—In this note, a counterexample to the main result given in [1] is constructed. We generalize a well-known result and obtain better bounds for the trace of two matrices.**

## I. Introduction

To obtain estimates of solutions of Lyapunov and Riccati equations which frequently occur in the stability analysis and optimal control design in linear control theory, many researchers have attempted to determine upper and lower bounds for the product of two matrices in terms of the trace of one matrix and the eigenvalues of the other. Recently, Baksalary and Puntanen [1] claimed that they had obtained a better estimate for the trace of the product of two matrices. The first purpose of this note is to point out that their main result is incorrect and a counterexample is presented. We also present a counterexample to a "tempting" conjecture and then generalize a result of Mori given in [2].

To facilitate the discussion, notions and previous results are presented next. Let $R_{n,n}$ denote the set of $n \times n$ real matrices; $S_n \subset R_{n,n}$ is the set of $n \times n$ real symmetric matrices, and $V_n \subset S_n$ is the set of $n \times n$ positive semidefinite matrices. Kleinman and Athans [3], in the context of design of suboptimal control systems, obtained that, for any $A \in V_n$ and $B \in V_n$

$$\lambda_n(A)\,\mathrm{tr}\,(B) \le \mathrm{tr}\,(AB) \le \lambda_1(A)\,\mathrm{tr}\,(B) \tag{1}$$

where $\lambda_i(A)$ is the $i$th largest eigenvalue of $A$. Wang *et al.* [4] proved that the inequality in (1) still holds if $A \in S_n$. Baksalary and Puntanen [1] claimed that "the two bounds in (1) hold also for any symmetric $B$, not necessarily nonnnegative definite" (Remark: from the context of [1], we can see that the nonnegative definiteness in [1] really means positive semidefiniteness). This is not true; for example, consider the matrices

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then $\mathrm{tr}\,(AB) = 1$, $\lambda_1(A) = 1$ and $\mathrm{tr}\,(B) = 0$, obviously, (1) is not valid for this case. Mori [2] generalized the above result to the case when $A$ is any real matrix.

*Theorem 1:* For any $A \in R_{n,n}$ and $B \in Y_n$, the following inequality holds

$$\lambda_n(\overline{A})\,\mathrm{tr}\,(B) \le \mathrm{tr}\,(AB) \le \lambda_1(\overline{A})\,\mathrm{tr}\,(B) \tag{2}$$

where $\overline{A} = (A + A')/2$. Baksalary and Puntanen [1] claimed that they obtained a better result than (2), especially for the case when $\overline{A}$ is indefinite.

*Proposition (Main Theorem in [1]):* Let $A \in R_{n,n}$ and $B \in V_n$, and let $\overline{A} = \frac{1}{2}(A + A')$. Further, let $\nu_*(\overline{A})$ and $\nu^*(\overline{A})$ denote the smallest and the largest negative eigenvalues of $\overline{A}$ if they exist, and $\nu_*(\overline{A}) = \nu^*(\overline{A}) = 0$ otherwise; let $\pi_*(\overline{A})$ and $\pi^*(\overline{A})$ denote the smallest and the largest positive eigenvalues of $\overline{A}$ if they exist, and $\pi_*(\overline{A}) = \pi^*(\overline{A}) = 0$ otherwise. Then

$$[\nu_*(\overline{A}) + \pi_*(\overline{A})]\,\mathrm{tr}\,(B) \le \mathrm{tr}\,(AB) \le [\nu^*(\overline{A}) + \pi^*(\overline{A})]\,\mathrm{tr}\,(B). \tag{3}$$

This result is incorrect; a counterexample is given next: Let

$$A = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

then $\mathrm{tr}\,(AB) = 17$, $\pi^*(\overline{A}) = 4$, $\nu^*(\overline{A}) = -3$, and $\mathrm{tr}\,(B) = 10$, thus the second inequality in (3) does not hold for this example. Replacing $A$ by $-A$, we can obtain a counterexample for the first inequality in (3). The problem with the main result in [1] occurs in the proof where the equalities $\lambda_n(A_1) = \pi_*(\overline{A})$, $\lambda_n(A_2) = -\nu^*(\overline{A})$ are used. According to the decomposition of $\overline{A}$, which is given, $\lambda_n(A_1) = 0$ and $\lambda_n(A_2) = 0$.

## II. Main Results

From Mori [2], we know that $\lambda_1(\overline{A}) = \mu_2(A)$, where $\mu_2(A)$ is the matrix measure of $A$ induced by the 2-norm. Also from [5], we know that $\max_i \mathrm{Re}\,\lambda_i(A) \le \mu_2(A)$, it is tempting to conjecture the following.

*Conjecture:* For any $A \in R_{n,n}$ and $B \in V_n$, the following inequality holds

$$\mathrm{tr}\,(AB) = \mathrm{tr}\,(\overline{A}B) \le \max_i \mathrm{Re}\,\lambda_i(A)\,\mathrm{tr}\,(B).$$

Unfortunately, this is not true. For example, let

$$A = \begin{pmatrix} -1 & 1 \\ 0 & -2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 10 & 5 \\ 5 & 3 \end{pmatrix}.$$

Then a simple computation gives $\mathrm{tr}\,(AB) = -11$, $\max_i \mathrm{Re}\,\lambda_i(A) = -1$, and $\mathrm{tr}\,(B) = 13$, and the conjecture is not true. One may attempt to obtain better estimates for $\mathrm{tr}\,(AB)$; the following result shows that (2) given by Mori in [2] is the best estimate in a certain sense.

*Theorem 2:* For $A \in R_{n,n}$ fixed, if $\alpha$ and $\beta$ are any numbers satisfying

$$\alpha\,\mathrm{tr}\,(B) \le \mathrm{tr}\,(AB) \le \beta\,\mathrm{tr}\,(B) \tag{4}$$

for any positive semi-definite matrix $B$, then $\alpha \le \lambda_n(\overline{A})$ and $\lambda_1(\overline{A}) \le \beta$, i.e., $\lambda_n(\overline{A})$ and $\lambda_1(\overline{A})$ are the tightest bounds for the inequality (4). Hence (2) is the best estimate of $\mathrm{tr}\,(AB)$ in this sense.

*Proof:* From (2), we know that $\lambda_1(\overline{A})$ satisfies (4), i.e., $\mathrm{tr}\,(AB) \le \lambda_1(\overline{A})\,\mathrm{tr}\,(B)$. It suffices to show that equality can be achieved by the proper choice of $B \in V_n$. Since $\overline{A}$ is symmetric, there exists a unitary matrix $U$, such that $U'\overline{A}U = \mathrm{diag}\,\{\lambda_1(\overline{A}), \cdots, \lambda_n(\overline{A})\}$. Then choose $B = U\,\mathrm{diag}\,\{1, 0, \cdots, 0\}U'$, we have

$$\begin{aligned}\mathrm{tr}\,(AB) &= \mathrm{tr}\,\overline{A}B = \mathrm{tr}\,(\overline{A}U\,\mathrm{diag}\,\{1, 0, \cdots, 0\}U') \\ &= \mathrm{tr}\,(U'\overline{A}U\,\mathrm{diag}\,\{1, 0, \cdots, 0\}) = \lambda_1(\overline{A}) = \lambda_1(\overline{A})\,\mathrm{tr}\,(B).\end{aligned}$$

From this we can conclude that $\lambda_1(\overline{A}) \le \beta$. A similar argument can be used to complete the proof.

*Remark:* Baksalary and Puntanen [1] also claimed that the bounds of [2] cannot be attained unless $\overline{A}$ is a definite matrix. From Theorem 2, we observe that this statement is not correct.

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All of the above inequalities are based on the assumption that $B$ is positive semidefinite. When $B$ is indefinite, (2) is no longer valid. The next result is similar to the result (2) for the case when $B$ is symmetric, but not necessarily positive semidefinite.

*Theorem 3:* For any matrix $A \in R_{n,n}$ and any symmetric $B \in S_n$, let $\overline{A} = (A + A')/2$. Then

$$\lambda_n(\overline{A})\,\mathrm{tr}\,(B) - \lambda_n(B)(n\lambda_n(\overline{A}) - \mathrm{tr}\,(A)) \leq \mathrm{tr}\,(AB) \leq \lambda_1(\overline{A})\,\mathrm{tr}\,(B) - \lambda_n(B)(n\lambda_1(\overline{A}) - \mathrm{tr}\,(A)). \quad (5)$$

*Proof:* Since $B$ is symmetric, there exists a real $\alpha$ such that $B + \alpha I$ is nonnegative definite. From (2), we obtain

$$\mathrm{tr}\,(A(B+\alpha I)) = \mathrm{tr}\,(\overline{A}(B+\alpha I)) \leq \lambda_1(\overline{A})\,\mathrm{tr}\,(B+\alpha I) = \lambda_1(\overline{A})\,\mathrm{tr}\,(B) + \alpha(n\lambda_1(\overline{A})).$$

From this we have

$$\mathrm{tr}\,(AB) \leq \lambda_1(\overline{A})\,\mathrm{tr}\,(B) + \alpha(n\lambda_1(\overline{A}) - \mathrm{tr}\,(A)). \quad (6)$$

Now let $\alpha = -\lambda_{\min}(B) = -\lambda_n(B)$ where $\lambda(X)$ represents any eigenvalue of the matrix $X$. Since $\lambda(B+\alpha I) = \lambda(B) - \lambda_{\min}(B) \geq 0$, so this choice of $\alpha$ satisfies the required condition. From (6) we obtain the second inequality in (5). In a similar fashion, we can prove the first inequality in (5). This concludes the proof.

*Remark:* For a positive definite matrix $B$, Theorem 3 improves the result by Mori. This can be seen from (5). When $B$ is positive definite, $\lambda_n(B) > 0$, and

$$n\lambda_1(\overline{A}) - \mathrm{tr}\,(A) = n\lambda_1(\overline{A}) - \mathrm{tr}\,(\overline{A}) = \sum_{i=1}^{n}(\lambda_1(\overline{A}) - \lambda_i(\overline{A})) \geq 0.$$

The upper bound given in (5) is tighter than the upper bound given in (2) for this case. A similar argument applies to the lower bound. Theorem 3 then generalizes the result of [2] and also gives a better estimate of the trace of the matrix product. This result can be used to obtain improved bounds for the solutions of Lyapunov and Riccati equations and will be presented in a subsequent paper.

Several examples are given next.

*Example 1:* Let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

From (2), we obtain

$$-4 \leq \mathrm{tr}\,(AB) \leq -4$$

but (5) yields

$$-2 \leq \mathrm{tr}\,(AB) \leq 2$$

which is an improved estimate of the $\mathrm{tr}\,(AB)$.

*Example 2:* Let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In this example, $B$ is indefinite and (2) is not valid for this case. The result (5) gives

$$-1 \leq \mathrm{tr}\,(AB) \leq 1.$$

Indeed, $\mathrm{tr}\,(AB) = 1$ which is equal to the upper bound!

## III. Conclusion

A counterexample to a recent result on the inequality of the trace of matrix product is given. We show that Mori's result given in [2] is the best possible in a certain sense. Then a generalization of Mori's result is presented, and improved bounds for the trace of product of an arbitrary matrix and a nonnegative definite matrix are obtained.


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# Properties of the Entire Set of Hurwitz Polynomials and Stability Analysis of Polynomial Families

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***Abstract*—It is proved in this note that all Hurwitz polynomials of order not less than *n* form two simply connected Borel cones in the polynomial parameter space. Based on this result, edge theorems for Hurwitz stability of general polyhedrons of polynomials and boundary theorems for Hurwitz stability of compact sets of polynomials are obtained. Both cases of families of polynomials with dependent and independent coefficients are considered. Different from the previous ones, our edge theorems and boundary theorems are applicable to both monic and nonmonic polynomical families and do not require the convexity or the connectivity of the set of polynomials. Moreover, our boundary theorem for families of polynomials with dependent coefficients does not require the coefficient dependency relation to be affine.**

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