that a major advantage of the feedback linearization technique is that the feedback (10) provides the open-loop control that exactly steers the system on a given path. This is normally of great help in motion planning problems.

REFERENCES

[1] C. Altafini, "Some properties of the general n-trailer," *Int. J. Control*, pp. 409–424, Mar. 2001.
[2] C. Altafini and P. Gutman, "Path following with reduced off-tracking for the n-trailer system," in *Proc. 37th Conf. Decision and Control*, Tampa, FL, Dec. 1998, pp. 3123–3128.
[3] C. C. de Wit, "Trends in mobile robot and vehicle control," in *Control Problems in Robotics*, K. Valavanis and B. Siciliano, Eds. London, U.K.: Springer-Verlag, 1998, Lecture notes in control and information sciences.
[4] C. C. de Wit, B. Siciliano, and B. Bastin, Eds., *Theory of Robot Control*. London, U.K.: Springer-Verlag, 1997.
[5] E. J. Davison, "The robust control of a servomechanism problem for linear time-invariant multivariable systems," *IEEE Trans. Automat. Contr.*, vol. AC-21, pp. 25–34, 1976.
[6] B. A. Francis and W. M. Wonham, "The internal model principle of control theory," *Automatica*, vol. 12, pp. 457–465, 1976.
[7] A. Isidori, *Nonlinear Control Systems*, 3rd ed. New York: Springer-Verlag, 1995.
[8] A. Isidori and C. I. Byrnes, "Output regulation of nonlinear systems," *IEEE Trans. Automat. Contr.*, vol. 35, pp. 131–140, Feb. 1990.
[9] A. D. Luca, G. Oriolo, and C. Samson, "Feedback control of a nonholonomic car-like robot," in *Robot Motion Planning and Control*, J. Laumond, Ed. London, U.K.: Springer-Verlag, 1998, Lecture notes in control and information sciences.
[10] A. Micaelli and C. Samson, "Trajectory tracking for unicycle-type and two-steering-wheels mobile robots," INRIA, Paris, France, Rep. 2097, 1993.
[11] P. Rouchon, M. Fliess, J. Levine, and P. Martin, "Flatness, motion planning and trailer systems," in *Proc. 32nd IEEE Conf. Decision and Control*, San Antonio, TX, Dec. 1993, pp. 2700–2075.
[12] M. Sampei, T. Tamura, T. Itoh, and M. Nakamichi, "Path tracking control of trailer-like mobile robot," in *Proc. IEEE/RSJ Int. Workshop Intelligent Robots and Systems*, Osaka, Japan, Nov. 1991, pp. 193–198.
[13] C. Samson, "Control of chained systems: Application to path-following and time-varying point stabilization of mobile robots," *IEEE Trans. Automat. Contr.*, vol. 40, pp. 64–77, Jan. 1995.
[14] O. Sørdalen and C. C. de Wit, "Path following and stabilization of a mobile robot," in *Proc. 2nd IFAC Symp. Nonlinear Control Systems Design*, Bordeaux, France, 1992, pp. 477–484.

On the Relationship Between the Sample Path and Moment Lyapunov Exponents for Jump Linear Systems

Yuguang Fang and Kenneth A. Loparo

***Abstract*—In this note, we study the relationship between the sample and moment Lyapunov exponents for jump linear systems. Using a large deviation theorem, a modified version of Arnold's formula for connecting sample path and moment Lyapunov exponents for continuous-time linear stochastic systems is extended to discrete-time jump linear systems. Sample path stability properties of linear stochastic systems are determined by the top Lyapunov exponent and relating sample and moment Lyapunov exponents may be useful for developing computationally efficient methods for determining the almost-sure (sample path) stability of linear stochastic systems.**

***Index Terms*—Finite-state Markov chain, large deviation, linear stochastic systems, Lyapunov exponents, moment Lyapunov exponents.**

I. INTRODUCTION

Determining the stability of a linear stochastic system is an important problem. In general, the most useful stability criteria involve sample-path or almost-sure stability of the system. Necessary and sufficient conditions for sample-path stability often require a difficult computation of the top Lyapunov exponent. Although moment stability calculations, e.g., stability of the mean or the second moment, only require the stability analysis of a deterministic system, the results might not be useful in practice. In particular, for a linear stochastic system, it is well known that second-moment stabilty implies sample-path stability, but often times second moment stability criteria are too conservative to be useful in applications [11]. In this note, we investigate extending Arnold's formula relating sample and moment Lyapunov exponents for continuous-time linear stochastic systems with diffusion-type processes to discrete-time linear systems with random jump processes. The eventual goal is to use the relationship between sample and moment Lyapunov exponents to develop computationally efficient procedures for evaluating the sample-path stability of discrete-time jump linear systems.

Consider the discrete-time system

$$x(k+1) = A_k x(k) \quad x(0) = x_0 \tag{1.1}$$

where $\{A_k\}_{k\geq 0}$ is a sequence of $Gl(d, R)$-valued random variables. Here, $Gl(d, R)$ is the general linear group of dimension d over the real field, R. Fixing coordinates, a representative element of $Gl(d, R)$ is a nonsingular $d \times d$ matrix over R. A sample trajectory of (1.1) is given by the action of a random matrix product on a point $x_0 \in R^d$. Our analysis is restricted to random matrices in $Gl(d, R)$ because of the importance of *regularity* of (1.1) [3].

The asymptotic behavior of sample trajectories of system (1.1) have been studied extensively by many researchers, most notably in the context of random matrix products (see [3]). Furstenberg and Kifer [6] considered the Lyapunov exponents and the corresponding subspace filtration of the state space, and obtained an integrability condition. Arnold [1] and Arnold *et al.* [2] have been studying moment Lyapunov exponents for linear stochastic systems and discovered a formula that

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connects the (top or largest) Lyapunov exponent and the (top) moment Lyapunov exponent for a class of linear stochastic systems where the form process is a diffusion. Motivated by this previous work, it is natural to ask whether this formula can be extended to linear discrete-time stochastic systems when the form process is a finte state Markov process. The conjecture is that if $A_k \in \{A(1), \ldots, A(N)\}$ and $\{A_k\}$ has certain properties, for example $\{A_k\}$ is an independent, identically distributed (iid) matrix sequence, or $\{A_k\}$ is an ergodic Markov process, then an Arnold-type formula also holds for (1.1). This note is in the spirit of our earlier work [9], where the relationship between the domain of almost sure stability and that of δ-moment stability was studied.

In this note, if $\{A_k\}$ is a random process governed by a finite state Markov chain and if a certain *regularity condition* is satisified, then we show that an Arnold-type formula connecting sample path and moment Lyapunov exponents holds. The proof of the main result of this note uses the large deviation theorem given in [8] and [9].

II. Relationship Connecting Sample and Moment Lyapunov Exponents

In this section, a modified version of Arnold's formula is derived to illustrate the relationship between the sample and moment Lyapunov exponents for the linear stochastic system (1.1). This reveals an important connection between sample stability and moment stability for this class of linear stochastic systems. Although the results presented in this note are restricted to a special class of linear stochastic systems, the results may hold in a more general setting. This will be discussed briefly later on in this note.

Consider the discrete-time system

$$x(k+1) = A(\sigma_k)x(k) \quad x(0) = x_0 \tag{2.1}$$

where $\{\sigma_k\}$ is a finite-state Markov chain form process with probability transition matrix $P = (p_{ij})_{N\times N}$. In what follows, it is assumed that $\{\sigma_k\}$ is irreducible with ergodic probability measure π and that the individual mode matrices $A(1), A(2), \ldots, A(N)$ are invertible $R^{d\times d}$ matrices.

Definition: The top (or largest) Lyapunov exponent and the top δ-moment Lyapunov exponent of system (2.1) are defined, respectively, as

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \log \|A(\sigma_n)\ldots A(\sigma_1)\|$$
$$g(\delta) = \lim_{n\to\infty} \frac{1}{n} \log E_\pi \|A(\sigma_n)\ldots A(\sigma_1)\|^\delta.$$

where $\|\cdot\|$ is any suitable matrix norm. The exponents λ and $g(\delta)$ are also given by

$$\lambda = \max_{x_0\neq 0} \lim_{k\to\infty} \frac{1}{k} \log \|x(k,x_0)\|$$
$$g(\delta) = \max_{x_0\neq 0} \lim_{k\to\infty} \frac{1}{k} \log E_\pi \|x(k,x_0)\|^\delta$$

where $x(k,x_0)$ is a sample solution of (2.1) with initial condition $x(0,x_0) = x_0$.

Theorem 2.1: For $\delta \geq 0$, the δ-moment Lyapunov exponent $g(\delta)$ is differentiable from the right at $\delta = 0$ and $g'(0+) = \lambda$.

Proof: From [9]

$$\lim_{n\to\infty} \frac{1}{n} \log \|A(\sigma_n)\ldots A(\sigma_1)\| = \lim_{n\to\infty} \frac{1}{n} E_\pi \log \|A(\sigma_n)\ldots A(\sigma_1)\| = \lambda$$

almost surely. Thus, for any $\varepsilon > 0$, there exists an $m > 0$ such that

$$E_\pi \log \|A(\sigma_m)\ldots A(\sigma_1)\| < m\lambda + m\varepsilon. \tag{2.2}$$

For any $p \in Z^+$

$$\frac{1}{pm} \log \|A(\sigma_{pm})\ldots A(\sigma_1)\| \leq \frac{1}{pm} \times \sum_{j=0}^{p-1} \log \left\|A(\sigma_{(j+1)m}\ldots A(\sigma_{jm+1})\right\|$$

and for any $\varepsilon > 0$ and $\varepsilon_1 > 0$ from (2.2)

$$\begin{aligned} P&\left(\frac{1}{pm} \log \|A(\sigma_{pm})\ldots A(\sigma_1)\| \geq \lambda + \varepsilon + \varepsilon_1\right) \\ &\leq P\left(\frac{1}{p} \sum_{j=0}^{p-1} \log \left\|A(\sigma_{(j+1)m})\ldots A(\sigma_{jm+1})\right\| \right. \\ &\qquad \left. \geq E_\pi \log \|A(\sigma_m)\ldots A(\sigma_1)\| + m\varepsilon_1\right) \end{aligned} \tag{2.3}$$

Using the large deviation theorem given in [9] , there exists r_1 satisfying $0 \leq r_1 < 1$ and an $M_1 > 0$ such that

$$\begin{aligned} P&\left(\frac{1}{p} \sum_{j=0}^{p-1} \log \left\|A(\sigma_{(j+1)m})\ldots A(\sigma_{jm+1})\right\| \right. \\ &\qquad \left. \geq E_\pi \log \|A(\sigma_m)\ldots A(\sigma_1)\| + m\varepsilon_1\right) \\ &\leq M_1 r_1^p = M_1 \left(r_1^{1/m}\right)^{pm}. \end{aligned}$$

From this and (2.3)

$$P\left(\frac{1}{pm} \log \|A(\sigma_{pm})\ldots A(\sigma_1)\| \geq \lambda + \varepsilon + \varepsilon_1\right) \leq M_1 \left(r_1^{1/m}\right)^{pm}. \tag{2.4}$$

Define the sets

$$\mathcal{A} = \left(\omega : \frac{1}{pm} \log \|A(\sigma_{pm})\ldots A(\sigma_1)\| \geq \lambda + \varepsilon + \varepsilon_1\right)$$
$$\mathcal{A}^c = \left(\omega : \frac{1}{pm} \log \|A(\sigma_{pm})\ldots A(\sigma_1)\| < \lambda + \varepsilon + \varepsilon_1\right).$$

Then, on $\mathcal{A}^c$

$$\|A(\sigma_{pm})\ldots A(\sigma_1)\| < e^{pm(\lambda+\varepsilon+\varepsilon_1)}$$

and on $\mathcal{A}$

$$\|A(\sigma_{pm})\ldots A(\sigma_1)\| \leq M^{pm}.$$

Thus

$$\begin{aligned} E_\pi \|A(\sigma_{pm})\ldots A(\sigma_1)\|^\delta &= \int_{\mathcal{A}} \|A(\sigma_{pm})\ldots A(\sigma_1)\|^\delta P(d\omega) \\ &\quad + \int_{\mathcal{A}^c} \|A(\sigma_{pm})\ldots A(\sigma_1)\|^\delta \\ &\quad \times P(d\omega) \\ &\leq M^{pm} P(\mathcal{A}) + e^{\delta pm(\lambda+\varepsilon+\varepsilon_1)} \\ &\leq M_1 \left(M^\delta r_1^{1/m}\right)^{pm} \\ &\quad + e^{\delta pm(\lambda+\varepsilon+\varepsilon_1)} \end{aligned} \tag{2.5}$$

where $\lim_{\delta\to 0} M^\delta r_1^{1/m} = r_1^{1/m} < 1$ implies there exists $\delta > 0$ sufficiently small and ρ satisfying $0 \leq \rho < 1$, such that $M^\delta r_1^{1/m} \leq \rho < 1$.

If $\lambda \geq 0$, then for sufficiently large p, from (2.5)

$$\begin{aligned}\frac{1}{pm}\log E_\pi \|A(\sigma_{pm})\dots A(\sigma_1)\|^\delta &\leq \frac{1}{pm}\log\left[M_1\rho^{pm} + e^{\delta pm(\lambda+\varepsilon+\varepsilon_1)}\right]\\ &\leq \frac{1}{pm}\log\left[1 + e^{\delta pm(\lambda+\varepsilon+\varepsilon_1)}\right]\\ &\leq \frac{1}{pm}\log\left[2e^{\delta pm(\lambda+\varepsilon+\varepsilon_1)}\right]\\ &= \frac{\log 2}{pm} + \delta(\lambda+\varepsilon+\varepsilon_1).\end{aligned}$$

Letting $p \to \infty$, $g(\delta)/\delta \leq \lambda + \varepsilon + \varepsilon_1$. From [9], $\lim_{\delta\downarrow 0+} g(\delta)/\delta = g'(0+)$ and $g'(0+) \geq \lambda$, and it follows that $g'(0+) = \lambda$ for $\lambda \geq 0$.

If $\lambda < 0$, select $\beta > -\lambda$ and define $B(j) = A(j)e^\beta$ ($j = 1, 2, \dots, N$). Then

$$\lim_{n\to\infty}\frac{1}{n}\log\|B(\sigma_n)\dots B(\sigma_1)\| = \lim_{n\to\infty}\frac{1}{n}\log\|A(\sigma_n)\dots A(\sigma_1)\| + \beta = \lambda + \beta > 0$$

and

$$G(\delta) \triangleq \lim_{n\to\infty}\frac{1}{n}\log E_\pi\|B(\sigma_n)\dots B(\sigma_1)\|^\delta = g(\delta) + \beta\delta.$$

Using the result for $\lambda \geq 0$, $G(\delta)$ is differentiable from the right at $\delta = 0$ and $G'(0+) = \lambda + \beta$. Because $G'(\delta) = g'(\delta) + \beta$, $g(\delta)$ is differentiable from the right at $\delta = 0$ and $g'(0+) = \lambda$. This completes the proof. □

In the proof of Theorem 2.1, the assumption that $A(1), A(2), \dots, A(N) \in Gl(d, R)$ is not used. If A_k is invertible for each k, then the definition of the δ-moment Lyapunov exponent $g(\delta)$ is well defined for all $\delta \in R$. Next, we study the differentiability of $g(\delta)$ at $\delta = 0$ for the invertible case. Before giving the main result, the concept of regularity is introduced.

Regularity Condition (RC): The system (2.1) is said to be regular if and only if

$$\lim_{k\to\infty}\frac{1}{k}\log\left\|A^{-1}(\sigma_1)A^{-1}(\sigma_2)\dots A^{-1}(\sigma_k)\right\| = -\lambda$$

where λ is the top Lyapunov exponent of system.

If system (2.1) is regular in the sense of [3], then the RC is also satisfied.

Theorem 2.2: Suppose that $A(1), \dots, A(N)$ are invertible, and that RC holds, then $g(\delta)$ is differentiable at $\delta = 0$ and $g'(0) = \lambda$.

Proof: From Theorem 2.1, $g(\delta)$ is differentiable from the right at $\delta = 0$ and $g'(0+) = \lambda$. Therefore, it is sufficient to prove that $g(\delta)$ is differentiable from the left and $g'(0-) = \lambda$. From [9] and RC

$$\lim_{n\to\infty}\frac{1}{n}\log\left\|A^{-1}(\sigma_1)\dots A^{-1}(\sigma_n)\right\| = \lim_{n\to\infty}\frac{1}{n}E_\pi\log\left\|A^{-1}(\sigma_1)\dots A^{-1}(\sigma_n)\right\| = -\lambda.$$

Then, for any $\varepsilon_1 > 0$, there exists an $m > 0$ such that

$$\begin{aligned}m(\lambda - \varepsilon_1) &\leq E_\pi\left\{-\log\left\|A^{-1}(\sigma_1)\dots A^{-1}(\sigma_m)\right\|\right\}\\ &\leq m(\lambda+\varepsilon_1).\end{aligned}$$

For any $\varepsilon_2 > 0$, let $d = \lambda - \varepsilon_1 - \varepsilon_2$. Then, from the inequality

$$\begin{aligned}1 = \|I\| &= \left\|A(\sigma_n)\dots A(\sigma_1)A^{-1}(\sigma_1)\dots A^{-1}(\sigma_n)\right\|\\ &\leq \|A(\sigma_n)\dots A(\sigma_1)\|\left\|A^{-1}(\sigma_1)\dots A^{-1}(\sigma_n)\right\|\end{aligned}$$

$$\frac{1}{pm}\log\|A(\sigma_{pm})\dots A(\sigma_1)\| \geq -\frac{1}{m}\left(\frac{1}{p}\sum_{j=0}^{p-1}\log\left\|A^{-1}(\sigma_{jm+1})\dots A^{-1}(\sigma_{(j+1)m})\right\|\right).$$

From this inequality

$$\begin{aligned}&P\left(\frac{1}{pm}\log\|A(\sigma_{pm})\dots A(\sigma_1)\| < d\right)\\ &\leq P\left(-\frac{1}{m}\left(\frac{1}{p}\sum_{j=0}^{p-1}\log\left\|A^{-1}(\sigma_{jm+1})\dots A^{-1}(\sigma_{(j+1)m})\right\|\right) < d\right)\\ &= P\left(\frac{1}{p}\sum_{j=0}^{p-1}\left\{-\log\left\|A^{-1}(\sigma_{jm+1})\dots A^{-1}(\sigma_{(j+1)m})\right\|\right\} < m(\lambda-\varepsilon_1) - m\varepsilon_2\right)\\ &\leq P\left(\frac{1}{p}\sum_{j=0}^{p-1}\left\{-\log\left\|A^{-1}(\sigma_{jm+1})\dots A^{-1}(\sigma_{(j+1)m})\right\|\right\} < E_\pi\left\{-\log\left\|A^{-1}(\sigma_1)\dots A^{-1}(\sigma_m)\right\|\right\} - m\varepsilon_2\right).\end{aligned}$$

Using the large deviation theorem from [9], there exists an $M_2 > 0$ and $\rho < 1$ such that

$$P\left(\frac{1}{pm}\log\|A(\sigma_{pm})\dots A(\sigma_1)\| < d\right) \leq M_2\rho^p. \tag{2.6}$$

Because $A(1), \dots, A(N)$ are invertible, there exists $M_1 > 0$ such that $\|A^{-1}(\sigma_k)\| \leq M_1$, and for $\delta < 0$

$$\begin{aligned}\|A(\sigma_n)\dots A(\sigma_1)\|^\delta &\leq \left(\left\|A^{-1}(\sigma_1)\dots A^{-1}(\sigma_n)\right\|^{-1}\right)^\delta\\ &\leq \left(\left\|A^{-1}(\sigma_1)\right\|\dots\left\|A^{-1}(\sigma_n)\right\|\right)^{-\delta}\\ &\leq M_1^{-n\delta}.\end{aligned}$$

Following the same procedure as in the proof of Theorem 2.1, for $\delta < 0$

$$\begin{aligned}E\|A(\sigma_{pm})\dots A(\sigma_1)\|^\delta &\leq M_1^{-pm\delta}P\left(\frac{1}{pm}\log\|A(\sigma_{pm})\dots A(\sigma_1)\| < d\right) + e^{dpm\delta}\\ &\leq M_2(M_1^{-m\delta}\rho)^p + e^{dpm\delta}.\end{aligned} \tag{2.7}$$

If $\lambda < 0$, then $d < 0$ and $\delta d > 0$ and for $|\delta|$ chosen sufficiently small and p chosen sufficiently large, $M_2(M_1^{-\delta}\rho)^p < 1$. (Note: $\lim_{\delta\to 0} M_1^{-m\delta}\rho = \rho < 1$). From (2.7)

$$E\|A(\sigma_{pm})\dots A(\sigma_1)\|^\delta \leq 1 + e^{dpm\delta} \leq 2e^{dpm\delta}$$

and

$$\begin{aligned}g(\delta) &= \lim_{p\to\infty}\frac{1}{pm}\log E\|A(\sigma_{pm})\dots A(\sigma_1)\|^\delta\\ &\leq \lim_{p\to\infty}\frac{1}{pm}\log\left(2e^{dpm\delta}\right) = d\delta.\end{aligned}$$

Because $\delta < 0$

$$\frac{g(\delta)}{\delta} \geq d = \lambda - \varepsilon_1 - \varepsilon_2.$$

Because ε_1 and ε_2 are arbitrary and $g(\delta)/\delta$ is a nondecreasing function [9], $g'(0-) \geq \lambda$ for $\lambda < 0$. From [9], $g(\delta)/\delta \leq \lambda$ for $\delta < 0$, then $g'(0-) = \lambda$ for $\lambda < 0$.

If $\lambda \geq 0$, choose $\beta > \lambda$ and consider the sequence $\{B(\sigma_n)\}$ where $B(\sigma_n) = A(\sigma_n)e^{-\beta}$. Then

$$\lim_{k\to\infty}\frac{1}{k}\log\|B(\sigma_k)\dots B(\sigma_1)\| = \lambda - \beta < 0$$

and

$$G(\delta) \triangleq \lim_{k\to\infty} \frac{1}{k} \log E \|B(\sigma_k)\dots B(\sigma_1)\|^\delta = g(\delta) - \beta\delta.$$

Here, $B(1), B(2), \dots, B(N)$ are invertible and (2.1) with the system matrix $B(\sigma_k)$ satisfies RC. From our previous result, $G'(0-) = \lambda - \beta$, therefore $g'(0-) = \lambda$. This completes the proof. □

The regularity condition can be difficult to check. The next result provides the same conclusion as Theorem 2.2 without requiring RC, and the proof follows directly from the the large deviation theorem given in [4].

Proposition 2.3: Suppose that $\{A_k\}$ is an iid matrix sequence in $Gl(d, R)$ and that there exists an $M > 0$ such that $\|A_k\| \le M$. If $E \log^+ \|A_1\| < +\infty$ (an integrability condition) holds, then the δ-moment Lyapunov exponent

$$g(\delta) = \lim_{k\to\infty} \frac{1}{k} \log E\|A_k \dots A_1\|^\delta$$

is differentiable from the right at $\delta = 0$ and $g'(0+)$ is equal to the top Lyapunov exponent

$$\lambda = \lim_{k\to\infty} \frac{1}{k} \log \|A_k \dots A_1\|.$$

□

Remark: Let $\{A_k\}$ be an iid random matrix sequence in $Gl(d, R)$ that satisfies the integrability condition

$$\int_\Omega \left[\log^+ \|A_1\| + \log^+ \|A_1^{-1}\|\right] P(d\omega) < +\infty.$$

If there exists an $M > 0$ and $\delta > 0$ such that

$$\lim_{k\to\infty} \int_{\mathcal{C}} e^{\delta \sum_{i=1}^n \log \|A_i\|} P(d\omega) = 0 \qquad (H)$$

where

$$\mathcal{C} = \left(\omega : \frac{1}{k} \sum_{i=1}^n \log \|A_i\| \ge M\right)$$

then $g(\delta)$ is differentiable from the right at $\delta = 0$ and $g'(0+) = \lambda$.

This result can be established from the following observations: For any $m > 0$ chosen in a manner similiar to the proof of Theorem 2.1, define the sets

$$\begin{aligned}
\mathcal{A} &= \left(\omega : \frac{1}{pm} \log \|A_{pm} \dots A_1\| \ge d\right) \\
\mathcal{A}^c &= \left(\omega : \frac{1}{pm} \log \|A_{pm} \dots A_1\| < d\right) \\
\mathcal{B} &= \left(\omega : \frac{1}{pm} \sum_{i=1}^{pm} \log \|A_i\| \ge M\right) \\
\mathcal{B}^c &= \left(\omega : \frac{1}{pm} \sum_{i=1}^{pm} \log \|A_i\| < M\right).
\end{aligned}$$

Then

$$\begin{aligned}
E \|A_{pm} \dots A_1\|^\delta &= \int_{\mathcal{A}} \|A_{pm} \dots A_1\|^\delta P(d\omega) \\
&\quad + \int_{\mathcal{A}^c} \|A_{pm} \dots A_1\|^\delta \\
&= \int_{\mathcal{A}\cap\mathcal{B}} \|A_{pm} \dots A_1\|^\delta P(d\omega) \\
&\quad + \int_{\mathcal{A}\cap\mathcal{B}^c} \|A_{pm} \dots A_1\|^\delta P(d\omega) \\
&\quad + \int_{\mathcal{A}^c} \|A_{pm} \dots A_1\|^\delta P(d\omega) \\
&\le \int_{\mathcal{B}} \|A_{pm} \dots A_1\|^\delta P(d\omega) \\
&\quad + \int_{\mathcal{A}\cap\mathcal{B}^c} \|A_{pm} \dots A_1\|^\delta P(d\omega) \\
&\quad + \int_{\mathcal{A}^c} \|A_{pm} \dots A_1\|^\delta P(d\omega) \\
&\le \int_{\mathcal{B}} e^{\delta \sum_{i=1}^{pm} \log \|A_i\|^\delta} P(d\omega) \\
&\quad + \int_{\mathcal{A}\cap\mathcal{B}^c} \|A_{pm} \dots A_1\|^\delta P(d\omega) \\
&\quad + \int_{\mathcal{A}^c} \|A_{pm} \dots A_1\|^\delta P(d\omega). \qquad (2.8)
\end{aligned}$$

If $\omega \in \mathcal{B}^c$, then $\|A_{pm} \dots A_1\| \le e^{pmM}$, and the technique used in the proof of Theorem 2.1 can be used to deal with the last two terms of (2.8). The first term in (2.8) is dealt with by hypothesis (H). Although we have not been able to prove that (H) follows from the iid and integrability properties of the matrix sequence $\{A_n\}$, the large deviation theorem of [4] suggests that this might be true. We will investigate this idea further in subsequent work.

Next, consider the continuous-time jump linear system

$$\dot{x}(t) = A(\sigma_t)x(t) \quad x(0) = x_0 \qquad (2.9)$$

where $\{\sigma_t\}$ is a finite-state Markov chain with infinitesimal generator $Q = (q_{ij})$. In what follows, $\{\sigma_t\}$ is an irreducible finite-state Markov chain with state space $S = \{1, 2, \dots, N\}$ and with ergodic measure π.

The (top) Lyapunov exponent and the (top) δ-moment Lyapunov exponent are defined, respectively, by

$$\begin{aligned}
\lambda &= \max_{x_0\neq 0} \overline{\lim_{t\to+\infty}} \frac{1}{t} \log \|x(t, x_0)\| \\
g(\delta) &= \max_{x_0\neq 0} \overline{\lim_{t\to+\infty}} \frac{1}{t} \log E_\pi \|x(t, x_0)\|^\delta
\end{aligned}$$

where $x(t, x_0)$ is the solution of (2.9) with initial condition $x(0, x_0) = x_0 \neq 0$.

The following result can be proved using techniques similar to those already presented and a sojourn description [11] of the random process σ_t.

Theorem 2.5: Given $\{\sigma_t\}$ a finite-state irreducible ergodic Markov chain, then $g(\delta)$ is differentiable from the right at $\delta = 0$ and $g'(0+) = \lambda$. Moreover, if (2.9) is regular [3], then $g(\delta)$ is differentiable at $\delta = 0$ and $g'(0) = \lambda$.

Remark: Using the sojourn description of the random process σ_t, the continuous-time jump linear system (2.9) is converted to a discrete-time jump linear system. It is not necessary to require that $A(1), \dots, A(N)$ in the continuous-time jump linear system are invertible because the exponential of these matrices is what appears in the discrete-time system.

Although our results have been restricted to the top sample and δ-moment Lyapunov exponents, using the random spectrum theory developed in [5] and [7], similar relationships can be obtained between the sample and δ-moment Lyapunov exponents. The subspaces L_i that define a filtration of the state space in the random spectrum theory are invariant subspaces of the state space of (2.1). These and related results, along with detailed proofs of Proposition 2.3 and Theorem 2.4, will be presented elsewhere.

III. Conclusion

This note has shown that Arnold's formula, which connects sample path stability and moment stability, as determined by the sign of the top sample and moment Lyapunov exponents of linear stochastic systems, can be extended to random matrix products (e.g., stochastic jump linear systems). It is also shown that the formula holds for a more general class of stochastic systems under a regularity type condition presented in this note.

REFERENCES

[1] L. Arnold, "A formula connecting sample and moment stability of linear stochastic systems," *SIAM J. Appl. Math.*, vol. 44, pp. 793–802, 1984.

[2] L. Arnold, W. Kleimann, and E. Oeljeklaus, "Lyapunov exponents of linear stochastic systems," in *Lecture Notes in Mathematics 1186*. New York: Springer-Verlag, 1986, pp. 85–125.

[3] L. Arnold and V. Wihstutz, Eds., *Lyapunov Exponents*. New York: Springer-Verlag, 1986, Lecture Notes in Mathematics 1186.

[4] R. S. Ellis, "Large deviations for a general class of random Vectors," *Ann. Probab.*, vol. 12, pp. 1–12, 1984.

[5] X. Feng, "Lyapunov Exponents and Stability of Stochastic Systems," Ph.D. dissertation, Dept. Syst. Eng., Case Western Reserve Univ., Cleveland, OH, 1990.

[6] H. Furstenberg and Y. Kifer, "Random matrix products and measures on projective spaces," *Israel J. Math.*, vol. 46, pp. 12–32, 1983.

[7] X. Feng and K. A. Loparo, "A nonrandom spectrum for products of random matrices and linear stochastic systems," *J. Math. Syst., Estimat., Control*, vol. 2, no. 3, pp. 323–338, 1992.

[8] ——, "Almost sure and δ-moment stability of jump linear systems," *Int. J. Control*, vol. 59, no. 5, pp. 1281–1307, 1994.

[9] ——, "Stability of discrete-time jump linear systems," *J. Math. Syst., Estimat., Control*, vol. 5, no. 3, pp. 275–321, 1995.

[10] R. Z. Hasminskii, *Stochastic Stability of Differential Equations*, The Netherlands: Sijthoff & Noordhoff, 1980.

[11] X. Feng, K. A. Loparo, Y. Ji, and H. Chizeck, "Stochastic stability properties of jump linear systems," *IEEE Trans. Automat. Contr.*, vol. 37, pp. 38–54, Jan. 1992.

Crossover Frequency Limitations in MIMO Nonminimum Phase Feedback Systems

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***Abstract*—This note investigates limitations and design tradeoffs of the closed-loop sensitivity/performance of linear-time-invariant nonminimum-phase uncertain multiple-input–multiple-output plants, with l inputs and m outputs, where $m \leq l$. It is shown that if rows $i_1, \ldots, i_k$ of the plant transfer function form a $k \times l$ nonminimum phase transfer matrix, and if the design is such that the sensitivity gain of $k - 1$ rows among the rows $i_1, \ldots, i_k$ of the closed-loop transfer function is low, then by necessity the sensitivity gain of the remaining row is high. This sensitivity constraint is quantified with the help of the crossover frequency restriction of a specially constructed single-input–single-output transfer function that includes the right half plane zeros and poles of the $k \times l$ transfer matrix.**

***Index Terms*—Feedback control, linear systems, multivariable systems, nonminimum phase, sensitivity.**

I. INTRODUCTION

It is well known that the benefit of feedback for nonminimum-phase (NMP) single-input–single-output (SISO) plants, as well as for NMP multiple-input–multiple-output (MIMO) plants, is limited. This NMP limitation appears when the plant has right-half plane (RHP) zeros, pure delay, or if the plant is sampled. Classic examples of NMP plants include flight control (aft δ_e control to elevation, and throttle command to elevation as measured close to the aircraft center of gravity) and the inverted pendulum.

The SISO case has been investigated widely in the literature. Reference [14] and [25] presented an optimal robust synthesis technique to design a feedback controller for an uncertain NMP plant to achieve a given closed-loop performance, providing the designer with insight into the tradeoffs between closed-loop performance and bandwidth, and also defining an implicit criterion for determining whether a solution exists. Reference [26] developed a criterion to estimate the maximum bandwidth of a sampled plant for given gain and phase margin, assuming an open loop of the ideal Bode characteristics form and using asymptotic approximations. Reference [12] extended this technique to stable plants with several RHP zeros, showing how to achieve a large open-loop gain in several frequency ranges, although there would always be some frequency ranges which are determined by the RHP zeros, in which the open-loop gain must be less than 0 dB. This known fact was proven in [6] and [7] showing that for NMP plants, a small sensitivity in one frequency range forces a large sensitivity in the complementary range. References [7] and [8] developed several constraints on the closed-loop sensitivity of NMP and/or unstable plants in the form of weighted integrals of the sensitivity on all frequencies or on a frequency range where the open-loop gain is much less than one. Reference [21] used their results to provide a bandwidth limitation on NMP and/or unstable plants. For crossover frequency limitations assuming a given slope of the open-loop amplitude around the crossover frequency, see [1].

The MIMO case is quite different from the SISO case. Reference [5] showed that the RHP transmission zeros of a MIMO plant are also transmission zeros of the plant output in any closed-loop stable structure. Reference [13] was the first to discuss the sensitivity of each element of the sensitivity transfer function of a MIMO plant showing that the MIMO quantitative feedback theory (QFT) design method can be applied to NMP plants where the cost is high sensitivity of at least all the elements of one row of the sensitivity transfer function, whereby the row can be chosen by the designer. This moving effect of the RHP zeros to a specific output was discussed in [27, Ch. 6.5]. Reference [30] showed explicitly the limitations of NMP plants in the LTR procedure, while [22] developed performance limitations of NMP MIMO systems measured by the cheap quadratic functional. The main result is a quantitative measure for the degree of difficulty in solving the servomechanism problem for NMP systems which is related to $\sum 1/\lambda_i$ where λ_i is a RHP zero of the plant. Reference [2] developed sensitivity integral relations by which the sensitivity tradeoff in different frequency ranges as a function of the RHP poles and zeros were extended to continuous-time MIMO plants, and in [4] extended to discrete-time MIMO plants. For a multivariable system with RHP zeros, [3] developed for its singular values an integral relation, akin to Bode's phase-gain relation, as well as an integral sensitivity relation. Reference [10] presented integral constraints, in the form of inequalities, for the sensitivity of unstable or nonminimum-phase MIMO feedback systems, giving insight into the sensitivity tradeoffs and the cost of decoupling in multivariable design. For time domain interpretations, see [19]. The discrete-time multivariable case was also discussed in [11], where analytic constraints for the sensitivity and mixed sensitivity functions were given using coprime factorization and state space representation. Reference [24] contains many of the integral results previously mentioned for SISO and MIMO continuous-time and discrete-time plants.

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