Stabilization of Continuous-Time Jump Linear Systems

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***Abstract*—In this paper, we investigate almost-sure and moment stabilization of continuous time jump linear systems with a finite-state Markov jump form process. We first clarify the concepts of δ-moment stabilizability, exponential δ-moment stabilizability, and stochastic δ-moment stabilizability. We then present results on the relationships among these concepts. Coupled Riccati equations that provide necessary and sufficient conditions for mean-square stabilization are given in detail, and an algorithm for solving the coupled Riccati equations is proposed. Moreover, we show that individual mode controllability implies almost-sure stabilizability, which is not true for other types of stabilizability. Finally, we present some testable sufficient conditions for δ-moment stabilizability and almost-sure stabilizability.**

***Index Terms*—Almost-sure stabilizability, coupled Riccati equations, δ-moment stabilizability, jump linear systems.**

I. Introduction

CONSIDER the continuous-time jump linear system in the form

$$\dot{x}(t) = A(\sigma(t))x(t) + B(\sigma(t))u(t) \tag{1.1}$$

or its discrete counterpart

$$x(t+1) = A(\sigma(t))x(t) + B(\sigma(t))u(t) \tag{1.2}$$

where $\sigma(t)$ is a finite-state random step process, usually a finite-state, time homogeneous, Markov process. The models (1.1) and (1.2) can be used to analyze the closed-loop stability of control systems with communication delays [1], [2] or the stability of control systems subject to abrupt phenomena such as component and interconnection failures [3]. The stability analysis of (1.1) or (1.2) is therefore very important in the design and analysis of a variety of control systems. Stability analysis of systems of this type can be traced back to the work of Rosenbloom [6] on moment stability properties. Bellman [4] was the first to study the moment stability of (1.2) with an i.i.d. form process using the Kronecker matrix product. Bergen [5] used a similar idea to study the moment stability properties of the continuous time system (1.1) with a piecewise constant form process $\{\sigma(t)\}$. Later, Bhuracha [7] used Bellman's idea developed in [4] to generalize Bergen's results and studied both asymptotic stability of the mean and exponential stability of the mean. Darkhovskii and Leibovich [8] investigated second moment stability of (1.1) where $\sigma(t)$ is a step process with the time intervals between jumps governed by an i.i.d. process. They extended Bhuracha's result and obtained necessary and sufficient conditions for second moment stability in terms of the Kronecker matrix product. Ladde and Šiljak [17] formulated the dynamic reliability problem for multiplexed control systems as a continuous-time jump linear system with a finite-state Markov form process and then derived a sufficient condition for second moment stability. Srichander and Walker [16] studied fault-tolerant control systems using a jump linear system model with a form process which is not directly observable to model the failure events. Kats and Krasovskii [9] and Bertram and Sarachik [10] used a stochastic version of Lyapunov's second method to study almost-sure stability and moment stability. Unfortunately, constructing an appropriate Lyapunov function is difficult and this is a well known disadvantage of Lyapunov's second method. Also, in many cases, the criteria obtained from this method are similar to moment stability criteria and are often too conservative for practical applications. For certain classes of systems, such as (1.1) or (1.2), it is possible to obtain testable stability conditions. Feng *et al.* [18] and Ji *et al.* [11], [12] used Lyapunov's second method to study the stability of (1.1) or (1.2) where $\{\sigma(t)\}$ is a finite-state Markov chain. Necessary and sufficient conditions are obtained for second moment stability and stabilizability of both continuous time (1.1) and discrete-time (1.2) jump linear systems. Fragoso and Costa [38], [39] have studied mean-square stability of continuous-time linear systems with Markovian jumping parameters. In [38], necessary and sufficient conditions are obtained when additive disturbances are included in the system. In [39], necessary and sufficient conditions are obtained using a linear matrix inequality (LMI) approach when only partial information on the mode parameter is available to the controller. In general, the development of second moment stability or stabilization criteria for jump linear systems involves the simultaneous solution of a system of coupled Riccati equations, [40]–[43]. In [44] and [45], the authors develop necessary and sufficient conditions for mean-square stabilization and consider the problem of obtaining a maximal solution of a system of coupled algebraic Riccati equations using an LMI approach. LMI techniques have proven to be useful in addressing computational issues associated with developing second moment stability criteria for jump linear systems.

As Kozin [13] pointed out, moment stability implies almost-sure stability under fairly general conditions, however, the converse is not true. In practical applications, almost-sure stability is usually the more desirable property because we can only observe a sample path of the system and the moment stability criteria can sometimes be too conservative to be practically useful.

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Although Lyapunov exponent techniques may provide necessary and sufficient conditions for almost-sure stability [14], [15], [18], [24]–[26], it is very difficult to compute the top Lyapunov exponent or to obtain good estimates of the top Lyapunov exponent for almost-sure stability. Testable conditions are difficult to obtain from this theory.

Arnold *et al.* [15] studied the relationship between the top Lyapunov exponent and the δ-moment top Lyapunov exponent for a diffusion process. Using a similar idea, Leizarowitz [25] obtained similar results for (1.1). A general conclusion was that δ-moment stability implies almost-sure stability. Thus, sufficient conditions for almost-sure stability can be obtained through δ-moment stability, which is one of the motivations for the study of δ-moment stability. There are many definitions for moment stability: δ-moment stability, exponential δ-moment stability and stochastic δ-moment stability. Ji *et al.* [11] proved that all second moment ($\delta = 2$) stability concepts are equivalent for (1.2). Feng *et al.* [18] showed that all the second moment stability concepts are equivalent for (1.1), and also proved that for a one dimensional system, the region for δ-moment stability is monotonically converging to the region for almost-sure stability as $\delta \downarrow 0^+$. This is tantamount to concluding that almost-sure stability is equivalent to δ-moment stability for sufficiently small δ. This is a significant result because the study of almost-sure stability can then be reduced to the study of δ-moment stability. In [20] and [22], we generalized the results reported in [18]. We showed that for (1.1) or (1.2) with a Markov form process, all δ-moment stability concepts are equivalent and they all imply almost-sure (sample) stability. We also showed that for sufficiently small $\delta > 0$, δ-moment stability and almost-sure exponential stability are equivalent. Henceforth, almost-sure stability can be inferred from δ-moment stability. Sufficient conditions for δ-moment stability and almost-sure stability were developed. A refined estimate of the δ-moment Lyapunov exponent given in [25] was also obtained.

This paper addresses the stabilization problem for a continuous time jump linear system. In Section II, some preliminaries and definitions are given. Section III is devoted to the δ-moment stabilization problem for $\delta > 0$, a necessary and sufficient condition for second moment stabilizability ($\delta = 2$) is given and some sufficient conditions for general $\delta > 0$ are presented. In Section IV, the almost-sure stabilization problem is studied and a relationship between almost-sure stabilizability and individual mode controllability (stabilizability) is illustrated along with some sufficient conditions for almost-sure stabilizability. Some illustrative examples are given in Section V.

II. Preliminaries and Definitions

We first establish some preliminaries for a finite-state Markov process $\{\sigma(t)\}$. Let $\underline{N} = \{1, 2, \ldots, N\}$. For all $i, j \in \underline{N}$, define

$$p_{ii} = 0$$
$$q_i = -q_{ii} = \sum_{l \neq i} q_{il}$$
$$p_{ij} = \frac{q_{ij}}{q_i}, \qquad (i \neq j).$$

Let $\{r_k;\ k = 1, 2, \ldots\}$ be the (discrete-time) Markov chain defined on the state-space $\underline{N}$ with the one-step transition matrix $(p_{ij})_{N \times N}$ and initial distribution ψ. This chain is referred to as the *embedded Markov chain* of $\{\sigma(t)\}$. We have the following sojourn time description of the process $\{\sigma(t)\}$ [34, p. 254].

Let τ_k, $k = 0, 1, \ldots$ be the successive sojourn times between jumps. Let $t_k = \sum_{l=0}^{k-1} \tau_l$ for $k = 1, 2, \ldots$ be the waiting time for the kth jump with $t_0 = 0$. Starting in state $\sigma(0) = i$, the process sojourns there for a duration of time that is exponentially distributed with parameter q_i. The process then jumps to the state $j \neq i$ with probability p_{ij}, the sojourn time in the state j is exponentially distributed with parameter q_j, and so on. The sequence of the states visited by the process $\{\sigma(t)\}$, denoted by $i_1, i_2, \ldots$ is the embedded Markov chain $\{r_k;\ k = 1, 2, \ldots\}$. Conditioning on $i_1, i_2, \ldots$, the successive sojourn times denoted by $\{\tau^{(i_k)}\}$ are independent exponentially distributed random variables with parameters q_{i_k}. Clearly, the joint process $\{(r_k, \tau_k)\colon k = 0, 1, \ldots\}$ is a time homogeneous Markov process that completely characterizes the form process $\{\sigma(t)\}$. The following notations will be used throughout this paper: $\mathcal{F}^n = \sigma\{(r_k, \tau_k)\colon 0 \leq k \leq n\}$ is the σ-algebra generated by $\{(r_k, \tau_k)\colon 0 \leq k \leq n\}$. For each $i \in \underline{N}$, e_i denotes the initial distribution of $\sigma(t)$ concentrated at the ith state. If $\sigma(t)$ has a single ergodic class, π denotes the unique invariant distribution of $\sigma(t)$. For a matrix B, $\lambda_i(B)$ denotes one of the eigenvalues of B, and $\lambda_{\max}(B) = \max_i(\mathrm{Re}\lambda_i(B))$ and $\lambda_{\min}(B) = \min_i(\mathrm{Re}\lambda_i(B))$ denote the largest and smallest real parts of the eigenvalues of B, respectively. $\det(A)$ denotes the determinant of a matrix A, $A \leq B$ ($A < B$) denotes that $B - A$ is a positive semidefinite (definite) matrix and $A \leq_e B$ ($A <_e B$) denotes an elementwise inequality.

Stochastic stability and stochastic stabilizability are always important issues in the design and analysis of stochastic control systems. Because the definitions for stochastic stability and stabilizability can be confusing, we next present the definitions that will be used in this paper.

Definition 2.1: Let Ξ denote the collection of probability measures on S and $\Psi \subset \Xi$ be a nonempty subset of Ξ. Consider the jump linear system

$$\dot{x}(t) = A(\sigma(t))x(t) \tag{2.1}$$

where $\{\sigma(t)\}$ is a finite-state Markov chain. For $\delta > 0$, (2.1) is said to be the following:

asymptotically δ-moment stable with respect to (w.r.t.) Ψ, if for any $x_0 \in R^n$ and any initial probability distribution $\psi \in \Psi$ of $\sigma(t)$

$$\lim_{t \to \infty} E\{\|x(t, x_0)\|^\delta\} = 0$$

where $x(t, x_0)$ is a sample solution of (2.1) initial from $x_0 \in \mathcal{R}^n$. If $\delta = 2$, we say that (2.1) is *asymptotically mean-square stable w.r.t.* Ψ; if $\delta = 1$, we say that (2.1) is *asymptotically mean stable w.r.t.* Ψ. If $\Psi = \Xi$, we say simply that (2.1) is *asymptotically δ-moment stable*. Similar statements apply to the following definitions.

Exponentially δ-moment stable with respect to Ψ, if for any $x_0 \in \mathcal{R}^n$ and any initial distribution $\psi \in \Xi$ of $\sigma(t)$, there exist constants $\alpha, \beta > 0$ independent of x_0 and ψ such that

$$E\{\|x(t, x_0)\|^\delta\} \le \alpha \|x_0\|^\delta e^{-\beta t} \qquad \forall t \ge 0.$$

Stochastically δ-moment stable with respect to Ψ, if for any $x_0 \in \mathcal{R}^n$ and any initial distribution $\psi \in \Psi$ of $\sigma(t)$

$$\int_{t=0}^{\infty} E\{\|x(t, x_0)\|^\delta\}\, dt < +\infty.$$

Almost surely (asymptotically) stable with respect to Ψ, if for any $x_0 \in \mathcal{R}^n$ and any initial distribution $\psi \in \Psi$ of $\sigma(t)$

$$P\left\{\lim_{t\to\infty} \|x(t, x_0)\| = 0\right\} = 1.$$

In the aforementioned definitions, the initial probability distribution of $\{\sigma(t)\}$ plays a very important role. The stochastic stability definitions as given can be interpreted in the context of robust stability, i.e., robustness to (Ψ-structured) uncertainty of the initial distributions of the form process. As the Markov process $(x(t), \sigma(t))$ is the state of the system and in practice, the initial probability distribution of the form process $\{\sigma(t)\}$ is usually not exactly known, this is a reasonable requirement. Also, stability with respect to a single initial distribution, say, the ergodic invariant distribution π, may not be a sufficient because a perturbation to π can destroy the stability of the system. The following example illustrates this point.

Example 2.1: Consider the one-dimensional (scalar) jump linear system (2.1), where $A(1) = a(1)$ and $A(2) = a(2) < 0$, assume that $\{\sigma(t)\}$ is a two state Markov chain with infinitesimal generator $Q = \left(\begin{smallmatrix} -1 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, and let P_ξ and E_ξ denote the probability measure and expectation with respect to the initial distribution ξ. It is easy to show that the unique invariant measure is $\pi = (0, 1)$. Thus, from

$$x(t) = x(0) \exp\left(\int_0^t A(\sigma(\tau))\, d\tau\right) = x_0 \exp\left(\int_0^t a(\sigma(\tau))\, d\tau\right)$$

we have for any $\delta > 0$

$$\begin{aligned} E_\pi \|x(t)\|^\delta &= \|x_0\|^\delta \exp\left(\int_0^t a(2)\, d\tau\right) \\ &= \|x_0\|^\delta \exp(\delta a(2) t) \xrightarrow{t\to\infty} 0. \end{aligned}$$

This implies that (2.1) is δ-moment stable with respect to the initial condition π (the invariant measure). However, if the initial distribution is $\xi = (1, 0)$, then

$$\begin{aligned} &E_\xi \|x(t)\|^\delta \\ &\quad \ge \|x_0\|^\delta \exp\left(\delta \int_0^t a(1)\, d\tau\right) \times P_\xi(\sigma(\tau) = 1,\ 0 \le \tau \le t) \\ &\quad = |x_0|^\delta e^{\delta a(1) t} P_\xi(\sigma(\tau) = 1,\ 0 \le \tau \le t | \sigma(0) = 1) \\ &\qquad \times P_\xi(\sigma(0) = 1) \\ &\quad = |x_0|^\delta e^{\delta a(1) t} e^{-t} = |x_0|^\delta e^{(\delta a(1) - 1)t}. \end{aligned}$$

Hence, if $\delta > 0$ and $a(1) > 0$ such that $\delta a(1) - 1 > 0$. Then from the previous computation, we can obtain

$$\lim_{t\to\infty} E_\xi \|x(t)\|^\delta = +\infty$$

which implies that the system is not δ-moment stable with the initial distribution $\xi = (1, 0)$.

This example shows that although transient states do not affect almost-sure stability, an obvious statement that is consistent with intuition, they do affect moment stability. One explanation is that when the system sojourns in a transient unstable state for too long, moment instability can occur.

The relationship among the stochastic stability concepts has been studied by Feng *et al.* [18] for $\delta = 2$, the mean-square stability case. Fang *et al.* [19] generalized this result to δ-moment stability for $\delta > 0$ for discrete-time jump linear systems. Fang [21] also extended such results to the continuous-time jump linear systems and obtained the following result.

Theorem 2.1: For any $\delta > 0$ and any system (2.1) with a finite-state Markov chain form process $\{\sigma(t)\}$, δ-moment stability, δ-moment exponential stability and stochastic stability are equivalent, and each implies almost-sure stability. □

Definition 2.2: The system (2.1) is said to be *absolutely stable* if it is stable in any sense in Definition 2.1 for any finite-state form process $\{\sigma(t)\}$.

Remark: This definition may be impractical in applications, however, if the system can be shown to be absolutely stable, then stochastic stability of the system is independent of the form process $(\sigma(t))$.

For stochastic stabilization, we give the following definition.

Definition 2.3: Consider the jump linear control system

$$\dot{x}(t) = A(\sigma(t))x(t) + B(\sigma(t))u(t). \tag{2.2}$$

If there exists a feedback control $u(t) = -K(\sigma(t))x(t)$ such that the resulting closed-loop control system is stochastically stable in the sense of Definition 2.1, then the control system (2.2) is said to be stochastically stabilizable in the corresponding sense. If the resulting closed-loop system is absolutely stable, then (2.2) is absolutely stabilizable. If the feedback control $K(\sigma(t)) = K$ is independent of the form process, then (2.2) is simultaneously stochastically stabilizable in the corresponding sense.

From Theorem 2.1, we can easily obtain the following result on the relationship among the previous stabilization concepts.

Corollary 2.2: For (2.2) with a finite-state Markov form process $\{\sigma(t)\}$ and with any $\delta > 0$, δ-moment stabilizability, δ-moment exponential stabilizability and stochastic δ-moment stabilizability are equivalent, and each implies almost-sure stabilizability. □

Remark: From now on, we will use δ-moment stabilizability to denote any one of the aforementioned three δ-moment stabilizability concepts.

It is easy to see that absolute stabilizability implies stochastic stabilizability in any sense, and simultaneous stochastic stabilizability implies stochastic stabilizability in the corresponding sense. However, absolute stabilizability is too conservative to be useful in applications. The simultaneous stochastic stabilizability problem has been studied in the current literature, however, simultaneous stabilizability is also too conservative. The next example is illustrative along this line.

Example 2.2: (δ-moment stabilizability does not imply simultaneous δ-moment stabilizability, and almost-sure stabilizability does not guarantee simultaneous almost-sure stabilizability).

Let $A(1) = a > 0$, $A(2) = b > 0$, $B(1) = 1$ and $B(2) = -1$. The form process $\{\sigma(t)\}$ has infinitesimal generator $Q = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$, with unique invariant measure $\pi = (0.5, 0.5)$. If we choose $K(1) = a+1$ and $K(2) = -b-1$, then $A(\sigma(t)) - B(\sigma(t))K(\sigma(t)) = -1$, hence the closed-loop system is deterministic and stable, and the jump linear system (2.2) is absolutely stabilizable, δ-moment stabilizable, and almost surely stabilizable.

However, we will show that the system cannot be simultaneously almost surely stabilized, and from Corollary 2.2 this implies that the system cannot be δ-moment stabilized. For any K, using the feedback control $u(t) = -Kx(t)$, the closed-loop system becomes

$$\dot{x}(t) = (A(\sigma(t)) - B(\sigma(t))K)x(t)$$

and its solution is given by

$$x(t) = x_0 \exp\left(\int_0^t (A(\sigma(\tau)) - B(\sigma(\tau))K)\, d\tau\right).$$

From this and the ergodic theorem, we obtain

$$\begin{aligned}
&\lim_{t\to\infty} \frac{1}{t} \log \frac{\|x(t)\|}{\|x_0\|} \\
&= \lim_{t\to\infty} \frac{1}{t} \int_0^t (A(\sigma(\tau)) - B(\sigma(\tau))K)\, d\tau \\
&= \pi_1(A(1) - B(1)K) + \pi_2(A(2) - B(2)K) \\
&= \tfrac{1}{2}(a - K + b + K) = \tfrac{1}{2}(a+b) > 0.
\end{aligned}$$

Therefore, (2.2) cannot be simultaneously almost surely stabilized.

III. δ-MOMENT STABILIZATION AND MEAN-SQUARE STABILIZATION

Mean-square (second moment) stabilizability problems have been studied by many researchers in the current literature. Ji *et al.* [12] reduced the stabilization problem to solving a coupled set of algebraic Riccati equations. Mariton [27], [28] applied homotopy theory to develop a numerical procedure for the mean-square ($\delta = 2$) stabilization problem. From Corollary 2.2, δ-moment stabilizability is equivalent to δ-moment stochastic stabilizability which involves a cost functional similar to linear quadratic optimal control systems design. Therefore, we may easily reduce the δ-moment stabilization problem to an appropriate optimal control problem. In this section, we obtain necessary and sufficient conditions for δ-moment stochastic stabilizability of a jump linear system when the mode process is directly observable. These results might be extended to include noise disturbances to the systems which have partial observations of the mode process using the results of [38] and [39].

Theorem 3.1: Given positive–definite matrices $Q(1), Q(2), \ldots, Q(N)$ and $R(1), R(2), \ldots, R(N)$, (2.2) is δ-moment stabilizable if and only if the following minimization problem:

$$\min J[u] = \int_0^\infty E(x^T(t)Q(\sigma(t))x(t) + u^T(t)R(\sigma(t))u(t))^{\delta/2}\, dt$$

subject to (2.2), has finite cost. In particular, (2.2) is δ-moment stabilizable if and only if the following optimal control problem:

$$\min J[u] = \int_0^\infty E(x^T(t)x(t) + u^T(t)u(t))^{\delta/2}\, dt$$

subject to (2.2), has finite cost.

Proof: Suppose that (2.2) is δ-moment stabilizable, then there exist matrices $K(1), K(2), \ldots, K(N)$ such that the system

$$\dot{x}(t) = (A(\sigma(t)) - B(\sigma(t))K(\sigma(t)))x(t) \tag{3.1}$$

is δ-moment stabilizable. Then, with the feedback control

$$u(t) = -K(\sigma(t))x(t)$$

it follows from Theorem 2.1 that

$$\int_0^\infty E\|x(t)\|^\delta\, dt = \int_0^\infty E(x^T(t)x(t))^{\delta/2}\, dt < \infty.$$

Thus, for the given control, we have

$$\begin{aligned}
J[u] &= \int_0^\infty E(x^T(t)Q(\sigma(t))x(t) \\
&\qquad + u^T(t)R(\sigma(t))u(t))^{\delta/2}\, dt \\
&= \int_0^\infty E(x^T(t)(Q(\sigma(t)) \\
&\qquad + K^T(\sigma(t))R(\sigma(t))K(\sigma(t)))x(t))^{\delta/2}\, dt \\
&\le M^{\delta/2} \int_0^\infty E(x^T(t)x(t))^{\delta/2}\, dt < \infty
\end{aligned}$$

where

$$M = \max_{1\le i\le N} \lambda_{\max}(Q(i) + K^T(i)R(i)K(i))$$

and the optimal control problem has finite cost.

Conversely, suppose that the given optimal control problem has a finite cost, then there exist matrices $K(1), K(2), \ldots, K(N)$ such that the solution of the closed-loop system (3.1) with the feedback control $u(t) = -K(\sigma(t))x(t)$ satisfies the following:

$$J[u] = \int_0^\infty E(x^T(t)Q(\sigma(t))x(t) + u^T(t)R(\sigma(t))u(t))^{\delta/2}\, dt < \infty$$

therefore, we have

$$\int_0^\infty E(x^T(t)x(t))^{\delta/2}\, dt \le M_1^{\delta/2} \int_0^\infty \times E(x^T(t)Q(\sigma(t))x(t))^{\delta/2}\, dt < \infty$$

where

$$M_1 = \max_{1\le i\le N} \frac{1}{\lambda_{\min}(Q(i))}.$$

From Theorem 2.1, (3.1) is δ-moment stable, hence (2.2) is δ-moment stabilizable. This completes the proof. □

It may seem that we have complicated the matter by reducing the stabilizability problem to an optimal control problem. However, the optimal control problem continues to be studied and many numerical algorithms have been developed in the literature. This is certainly the case for second moment stabilizability.

Corollary 3.1 [12], [37]: Given positive–definite matrices $Q(1), Q(2), \ldots, Q(N)$ and $R(1), R(2), \ldots, R(N)$, then (2.2) is mean-square (second moment) stabilizable if and only if the following coupled system of algebraic Riccati equations:

$$A^T(i)P(i) + P(i)A(i) - P(i)B(i)R^{-1}(i)B^T(i)P(i) + \sum_{i=1}^{N} q_{ij}P(j) = -Q(i), \qquad (i = 1, 2, \ldots, N) \quad (3.2)$$

has a positive–definite solution $P(1), P(2), \ldots, P(N)$.

In particular, (2.2) is mean-square stabilizable if and only if the following coupled system of algebraic Riccati equations:

$$A^T(i)P(i) + P(i)A(i) - P(i)B(i)B^T(i)P(i) + \sum_{i=1}^{N} q_{ij}P(j) = -I, \qquad (i = 1, 2, \ldots, N) \quad (3.3)$$

has a positive–definite solution $P(1), P(2), \ldots, P(N)$. □

Corollary 3.2: If (2.2) is mean-square stabilizable, then there exist positive–definite matrices $P(1), P(2), \ldots, P(N)$ such that

$$A(i) - B(i)B^T(i)P(i) - \tfrac{1}{2}q_iI \qquad (i = 1, 2, \ldots, N)$$

are stable.

Proof: Suppose that (2.2) is mean-square stabilizable, from Corollary 3.1, there exist positive–definite matrices $P(1), P(2), \ldots, P(N)$ such that (3.3) holds. From (3.3), we can easily obtain

$$\begin{aligned}&(A(i) - B(i)B^T(i)P(i) - \tfrac{1}{2}q_iI)^TP(i)\\ &\quad + P(i)(A(i) - B(i)B^T(i)P(i) - \tfrac{1}{2}q_iI)\\ &= -I - \sum_{j\neq i} q_{ij}P(j) - P(i)B(i)B^T(i)P(i).\end{aligned}$$

Because $P(i)$ and $I + \sum_{j=1}^{N} q_{ij}P(j) + P(i)B(i)B^T(i)P(i)$ are positive definite, from Lyapunov theory, we conclude that $(A(i) - B(i)B^T(i)P(i) - (1/2)q_iI)$ is stable. □

It is obvious that the mean-square stabilizability problem is equivalent to the existence of a positive–definite solution of the coupled system of Riccati equations, this does not, however, reduce the complexity of the problem considerably. For a linear time-invariant system, controllability implies stabilizability. One natural question to ask is: does individual mode controllability imply δ-moment stabilizability? For $\delta = 2$, Corollary 3.2 can be used to construct a simple example to show that the answer to this question is no.

Example 3.1: (*Individual mode controllability does not imply mean-square stabilizability*). Let

$$A(1) = \begin{pmatrix} 0.5 & 10 \\ 0 & 0.5 \end{pmatrix} \quad B(1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$A(2) = \begin{pmatrix} 0.5 & 0 \\ 10 & 0.5 \end{pmatrix} \quad B(2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$Q = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

where Q is the infinitesimal generator of the two-state Markov chain $\sigma(t)$. It is obvious that $(A(1), B(1))$ and $(A(2), B(2))$ are controllable, hence, (2.2) is individual mode controllable. However, for any positive matrix $P = \left(\begin{smallmatrix} p_{11} & p_{12} \\ p_{21} & p_{22}\end{smallmatrix}\right)$, we have

$$A(1) - B(1)B^T(1)P - \tfrac{1}{2}q_1I = \begin{pmatrix} -p_{11} & 10 - p_{12} \\ 0 & 0 \end{pmatrix}$$

which is not stable. From Corollary 3.2, we can conclude that (2.2) is not mean-square stabilizable. Notice also that the form process $\sigma(t)$ with infinitesimal generator Q is an ergodic Markov chain!

From Corollary 3.1, even though (2.2) is individual mode controllable, the infinite-horizon linear quadratic optimal control problem does not have a solution.

The mean-square stabilizability problem has been reduced to the solvability of a coupled Riccati equation (3.2) or (3.3) and it is very difficult to solve these equations analytically. Wonham [37] gave a recursive procedure for solving a coupled system of Riccati equations. Wonham's algorithm involved integration over an infinite horizon, which makes the algorithm computationally impractical. In order to obtain some qualitative properties about the solution of the coupled systems of Riccati equations, we first consider coupled Lyapunov equations. Coupled Lyapunov equations play a key role in the study of mean-square stability (see [21] for details).

Consider the coupled system of Lyapunov equations

$$A^T(i)P(i) + P(i)A(i) + \sum_{j=1}^{N} q_{ij}P(j) = -Q(i), \qquad i \in \underline{N} \quad (3.4)$$

where $\underline{N} = \{1, 2, \ldots, N\}$. Let $\mathrm{vec}(X)$ denote the column vector expansion of a matrix X, $\otimes$ denotes the Kronecker product and $\oplus$ denotes the Kronecker sum, i.e., $A \oplus B = A \otimes I + B \otimes I$ (see [35]). We have the following.

Theorem 3.2: For any matrices $Q(1), Q(2), \ldots, Q(N)$, (3.4) has a unique solution if and only if the matrix F, shown at the bottom of the next page, is nonsingular, where Q is the infinitesimal generator of the finite-state Markov chain $\{\sigma(t)\}$.

Proof: From (3.4), applying the vector expansion operator to both sides of (3.4) and using the property $\mathrm{vec}(AXB) = (B^T \otimes A)\mathrm{vec}(X)$ (refer to [35]), we obtain

$$(I \otimes A^T(i))\mathrm{vec}(P(i)) + (A^T(i) \otimes I)\mathrm{vec}(P(i)) + \sum_{j=1}^{N} q_{ij}\mathrm{vec}(P(j)) = -\mathrm{vec}(Q(i))$$

hence, we obtain

$$F \begin{pmatrix} \mathrm{vec}(P(1)) \\ \mathrm{vec}(P(2)) \\ \vdots \\ \mathrm{vec}(P(N)) \end{pmatrix} = -\begin{pmatrix} \mathrm{vec}(Q(1)) \\ \mathrm{vec}(Q(2)) \\ \vdots \\ \mathrm{vec}(Q(N)) \end{pmatrix}.$$

From this, we conclude that (3.4) has a unique solution if and only if F is nonsingular. □

From the result in [18] and Theorem 3.2, we easily obtain the following result, also see [22].

Theorem 3.3: The jump linear system $\dot{x}(t) = A(\sigma(t))x(t)$ is mean-square stable if and only if for any positive–definite matrices $Q(1), Q(2), \ldots, Q(N)$, the coupled system of Lyapunov equations (3.4) has a positive–definite solution, equivalently, if and only if F is Hurwitz stable. □

Corollary 3.3: Given positive–definite matrices $Q(1), Q(2), \ldots, Q(N)$, then the coupled system of Lyapunov equations (3.4) has a positive–definite solution $P(1), P(2), \ldots, P(N)$ if and only if F is Hurwitz. □

Corollary 3.4: Given positive–semidefinite matrices $Q(1), Q(2), \ldots, Q(N)$, if F is Hurwitz stable, then the coupled system of Lyapunov equations (3.4) has a positive–semidefinite solution $P(1), P(2), \ldots, P(N)$.

Proof: Since $Q(i) \geq 0$, for any $\beta > 0$, $Q(i) + \beta I > 0$. If F is Hurwitz stable, then from Corollary 3.3, the coupled Lyapunov equation

$$A^T(i)P(i) + P(i)A(i) + \sum_{j=1}^{N} q_{ij}P(j) = -(Q(i) + \beta I), \quad i \in \underline{N} \tag{3.5}$$

has a unique solution, say, $P(1, \beta), \ldots, P(N, \beta)$, and

$$\begin{pmatrix} \text{vec}(P(1, \beta)) \\ \text{vec}(P(2, \beta)) \\ \vdots \\ \text{vec}(P(N, \beta)) \end{pmatrix} = -F^{-1} \begin{pmatrix} \text{vec}(Q(1) + \beta I) \\ \text{vec}(Q(2) + \beta I) \\ \vdots \\ \text{vec}(Q(N) + \beta I) \end{pmatrix}. \tag{3.6}$$

Because F^{-1} is a constant matrix, the right hand side of (3.6) is a continuous function of β, and so is the left-hand side of (3.6). Thus $P(i, \beta)$ and $\lambda_{\min}(P(i, \beta))$ are continuous functions of β. From $\lambda_{\min}(P(i, \beta)) > 0$, we obtain $\lambda_{\min}(P(i, 0)) \geq 0$, i.e., $P(i, 0)$ is a positive–semidefinite solution of (3.4). This completes the proof. □

Next, we study the properties of the solutions of a coupled system of Riccati equations.

Theorem 3.4: If the coupled system of Riccati equations (3.3) has a positive–definite solution, then it is unique. That is, (3.3) has at most one positive–definite solution.

Proof: Let $P(i)$ and $\overline{P}(i)$ ($i \in \underline{N}$) be two positive–definite solutions of (3.3), let $K(i) = -B^T(i)P(i)$ and $\overline{K}(i) = -B^T(i)\overline{P}(i)$, then from (3.3), we have

$$(A(i) - B(i)K(i))^T P(i) + P(i)(A(i) - B(i)K(i)) + K^T(i)K(i) + \sum_{j=1}^{N} q_{ij}P(j) = -I \tag{3.7}$$

$$\left(A(i) - B(i)\overline{K}(i)\right)^T \overline{P}(i) + \overline{P}(i)\left(A(i) - B(i)\overline{K}(i)\right) + \overline{K}^T(i)\overline{K}(i) + \sum_{j=1}^{N} q_{ij}\overline{P}(j) = -I. \tag{3.8}$$

Let $E(i) = P(i) - \overline{P}(i)$, subtracting (3.7) from (3.8) and using the following identity:

$$\begin{aligned}(A(i) - B(i)K(i))^T P(i) &+ P(i)(A(i) - B(i)K(i)) + K^T(i)K(i) \\ &= (A(i) - B(i)\overline{K}(i))^T P(i) + P(i)(A(i) - B(i)\overline{K}(i)) \\ &\quad + \overline{K}(i)\overline{K}(i) - (K(i) - \overline{K}(i))^T(K(i) - \overline{K}(i))\end{aligned}$$

we can obtain

$$\left(A(i) - B(i)\overline{K}(i)\right)^T E(i) + E(i)(A(i) - B(i)\overline{K}(i) + \sum_{j=1}^{N} q_{ij}E(j) = \left(K(i) - \overline{K}(i)\right)^T(K(i) - \overline{K}(i)\right). \tag{3.9}$$

Because $I + \overline{K}^T(i)\overline{K}(i)$ and $\overline{P}(i)$ are positive–definite matrices, from (3.8) and Corollary 3.3 the matrix

$$\begin{aligned}\overline{F} = \text{diag}\Big\{&\left(A(1) - B(1)\overline{K}(1)\right)^T \\ &\oplus \left(A(1) - B(1)\overline{K}(1)\right), \ldots, \\ &\left(A(N) - B(N)\overline{K}(N)\right)^T \\ &\oplus \left(A(N) - B(N)\overline{K}(N)\right)\Big\} + Q \otimes I\end{aligned}$$

is Hurwitz. From (3.9) and Corollary 3.4, $E(i) \leq 0$, i.e., $P(i) \leq \overline{P}(i)$. Switching the roles of $P(i)$ and $\overline{P}(i)$, we conclude that $\overline{P}(i) \leq P(i)$, hence, $P(i) = \overline{P}(i)$. This concludes the proof. □

Next, a recursive procedure for solving the coupled system of Riccati equations (3.3) is given.

Algorithm:

Step 1) Suppose that there are no positive–definite matrices $P(1), P(2), \ldots, P(N)$ such that the matrices

$$A(i) - B(i)B^T(i)P(i) - \tfrac{1}{2}q_i I, \qquad i \in \underline{N} \tag{A1}$$

are all stable, then (2.2) is not mean-square stabilizable, and the algorithm terminates. Otherwise, we find a set of such positive–definite matrices, denoted by $P_0(i)$ ($i \in \underline{N}$) and let $P(i) = P_0(i)$ in (A1).

Step 2) Suppose that at the kth step in the algorithm we have found positive–definite matrices $P_k(i)$ ($i \in \underline{N}$),

---

$$\begin{aligned}F &= \begin{pmatrix} A^T(1) \oplus A^T(1) - q_1 I & q_{12}I & \cdots & q_{1N}I \\ q_{21}I & A^T(2) \oplus A^T(2) - q_2 I & \cdots & q_{2N}I \\ \vdots & \vdots & \ddots & \vdots \\ q_{N1}I & q_{N2}I & \cdots & A^T(N) \oplus A^T(N) - q_N I \end{pmatrix} \\ &= \text{diag}\{A^T(1) \oplus A^T(1), \ldots, A^T(N) \oplus A^T(N)\} + Q \otimes I\end{aligned}$$

the following Lyapunov equations are solved for the positive–definite matrices $P(i)$ $(i \in \underline{N})$:

$$\left(A(i) - B(i)B^T(i)P_k(i) - \tfrac{1}{2}\, q_i I\right)^T P(i) + P(i)\left(A(i) - B(i)B^T(i)P_k(i) - \tfrac{1}{2}\, q_i I\right) = -I - \sum_{j \neq i} q_{ij} P_k(j) - P_k(i)B(i)B^T(i)P_k(i).$$

Let $P_{k+1}(i) = P(i)$ $(i \in \underline{N})$.

Step 3) Return to Step 2) with $k \longrightarrow k+1$ and solve for $P_{k+2}(i)$ $(i \in \underline{N})$.

In order to establish the validity of this algorithm, we need to first show that in Step 2) a positive–definite solution $P(i)$ exists. This requires showing that if the solution $P_k(i)$ at the kth iteration is positive–definite, then $A(i) - B(i)B^T(i)P_k(i) - 0.5 q_i I$ is stable. This then guarantees the existence of a positive–definite solution $P_{k+1}(i)$. In fact, suppose that $P_k(i)$ is the positive–definite solution at the kth iteration, i.e.,

$$\left(A(i) - B(i)B^T(i)P_{k-1}(i) - \tfrac{1}{2}\, q_i I\right)^T P_k(i) + P_k(i)\left(A(i) - B(i)B^T(i)P_{k-1}(i) - \tfrac{1}{2}\, q_i I\right) = -I - \sum_{j \neq i} q_{ij} P_{k-1}(j) - P_{k-1}(i)B(i)B^T(i)P_{k-1}(i)$$

from which we obtain the following:

$$\left(A(i) - B(i)B^T(i)P_k(i) - \tfrac{1}{2}\, q_i I\right)^T P_k(i) + P_k(i)\left(A(i) - B(i)B^T(i)P_k(i) - \tfrac{1}{2}\, q_i I\right) = -\left(I + \sum_{j \neq i} q_{ij} P_{k-1}(j) + P_k(i)B(i)B^T(i)P_k(i) + \Delta P_k(i)B(i)B^T(i)\Delta P_k(i)\right) \quad (3.10)$$

where $\Delta P_k(i) = P_k(i) - P_{k-1}(i)$. Because $P_k(i)$ is assumed to be positive–definite, from (3.10) the matrix $(A(i) - B(i)B^T(i)P_k(i) - (1/2)q_i I)$ is stable and the Lyapunov equation

$$\left(A(i) - B(i)B^T(i)P_k(i) - \tfrac{1}{2}\, q_i I\right)^T P_{k+1}(i) + P_{k+1}(i)\left(A(i) - B(i)B^T(i)P_k(i) - \tfrac{1}{2}\, q_i I\right) = -I - \sum_{j \neq i} q_{ij} P_k(j) - P_k B(i)B^T(i)P_k(i) \quad (3.11)$$

has a positive–definite solution $P_{k+1}(i)$.

It is easy to see that if the algorithm converges, then the limit of $P_k(i)$ is the solution of (3.3) and (2.2) is mean-square stabilizable. The next question is when does the algorithm converge?

From Corollary 3.2, if system (2.2) is mean-square stabilizable, then $(A(i) - (1/2)q_i I,\, B(i))$ is stabilizable, and for any positive–definite matrices $Q(1), \ldots, Q(N)$, the Riccati equations in (3.12) have unique positive–definite solutions $P_0(1), \ldots, P_0(N)$:

$$\left(A(i) - \tfrac{1}{2}\, q_i I\right)^T P_0(i) + P_0(i)\left(A(i) - \tfrac{1}{2}\, q_i I\right) - P_0(i)B(i)B^T(i)P_0(i) = -Q(i), \qquad i \in \underline{N}. \quad (3.12)$$

Using the solutions of (3.12) to initialize the algorithm, we obtain the following result.

Theorem 3.5: If there exists positive–definite matrices $Q(1), \ldots, Q(N)$ such that the positive–definite solution $P_0(1), P_0(2), \ldots, P_0(N)$ of (3.12) satisfies

$$\int_0^\infty e^{(A(i)-B(i)B^T(i)P_0(i)-(1/2)q_i I)^T t} \times \left\{I + \sum_{j \neq i} q_{ij} P_0(j) - Q(i)\right\} \times e^{(A(i)-B(i)B^T(i)P_0(i)-(1/2)q_i I)t}\, dt \le 0$$

for any $i \in \underline{N}$, then the algorithm initialized with this solution converges and the coupled system of Riccati equations (3.3) has a unique positive–definite solution and (2.2) is mean-square stabilizable.

Proof: We only need to prove that the algorithm converges. Subtracting (3.10) from (3.11), we obtain $(k > 0)$

$$\left(A(i) - B(i)B^T(i)P_k(i) - \tfrac{1}{2}\, q_i I\right)^T \Delta P_{k+1}(i) + \Delta P_{k+1}(i)\left(A(i) - B(i)B^T(i)P_k(i) - \tfrac{1}{2}\, q_i I\right) = -\sum_{j \neq i} q_{ij} \Delta P_k(j) + \Delta P_k B(i)B^T(i)\Delta P_k(i), \quad (3.13)$$

from which we arrive at

$$\Delta P_{k+1}(i) = \int_0^\infty e^{(A(i)-B(i)B^T(i)P_k(i)-(1/2)q_i I)^T t} \times \left\{\sum_{j \neq i} q_{ij} \Delta P_k(j) - \Delta P_k(i)B(i)B^T(i)\Delta P_k(i)\right\} \times e^{(A(i)-B(i)B^T(i)P_k(i)-(1/2)q_i I)t}\, dt \le \int_0^\infty e^{(A(i)-B(i)B^T(i)P_k(i)-(1/2)q_i I)^T t} \times \left\{\sum_{j \neq i} q_{ij} \Delta P_k(j)\right\} \times e^{(A(i)-B(i)B^T(i)P_k(i)-(1/2)q_i I)t}\, dt. \quad (3.14)$$

Thus, if $P_0(i)$ is the positive–definite solution of (3.12), then $\Delta P_1(i) \le 0$ $(i \in \underline{N})$. Applying induction to (3.13), $\Delta P_{k+1}(i) \le 0$, i.e., $0 < P_{k+1}(i) \le P_k(i)$. This implies that the sequence $P_k(i)$ converges, and the algorithm is convergent. □

Remark: The condition given in Theorem 3.5 plays a role similar to condition [37, (6.12)].

We have only discussed the mean-square stabilization problem, which has been a central topic in the literature. There are essentially no results for δ-moment stabilization for arbitrary $\delta > 0$. Even the mean-square stabilization results are complicated and difficult to use. In [19] and [21], some δ-moment stability criteria are given, these can be used to study the δ-moment stabilization problem. This approach is studied next.

We first give a result for mean-square stabilization.

Theorem 3.6: The system (2.2) is mean-square stabilizable if and only if there exist matrices $K(1), \ldots, K(N)$ such that the matrix

$$H = \mathrm{diag}\{(A(1)-B(1)K(1))^T \oplus (A(1)-B(1)K(1))^T, \ldots, (A(N)-B(N)K(N))^T \oplus (A(N)-B(N)K(N))^T\} + Q \otimes I$$

is Hurwitz (I is an identity matrix of appropriate dimension).

Proof: Follows from the mean-square stability result obtained in [19] and [21]. □

Thus, the mean-square stabilization problem requires choosing feedback matrices to stabilize one "larger" matrix. Mariton [27] applied homotopy theory to the numerical computation of the feedback matrices, $K(1), \ldots, K(N)$. Next, we present some similar results for δ-moment stabilizability. This requires the concept of a matrix measure. Let $|x|$ denote a vector norm of x on C^n, and $||A||$ is the induced matrix norm of A given the vector norm $|\cdot|$. The matrix measure of A, $\mu(A)$, is defined as

$$\mu(A) \triangleq \lim_{\theta \downarrow 0^+} \frac{||I+\theta A|| - 1}{\theta}$$

where I is identity matrix. Properties of matrix measure can be found in [31]–[33]. For general $\delta > 0$, we have the following result.

Theorem 3.7: Let $\mu(\cdot)$ be an induced matrix measure [32]. Define

$$\overline{A}(i) = A(i) - B(i)K(i) + \frac{\delta-2}{2}\,\mu(A(i) - B(i)K(i))I$$
$$\underline{A}(i) = A(i) - B(i)K(i) - \frac{\delta-2}{2}\,\mu(-A(i) + B(i)K(i))I, \quad (i \in \underline{N}).$$

Define

$$H(\delta) = \begin{cases} \mathrm{diag}\left\{\overline{A}^T(1) \oplus \overline{A}(1), \ldots, \overline{A}^T(N) \oplus \overline{A}^T(N)\right\} + Q \otimes I, & \delta \geq 2; \\ \mathrm{diag}\left\{\underline{A}^T(1) \oplus \underline{A}(1), \ldots, \underline{A}^T(N) \oplus \underline{A}^T(N)\right\} + Q \otimes I, & \delta < 2. \end{cases}$$

If there exist matrices $K(1), \ldots, K(N)$ such that $H(\delta)$ is Hurwitz, then (2.2) is δ-moment stabilizable.

Proof: From [21], the δ-moment top Lyapunov exponent of (2.2) with the feedback control $u(t) = -K(\sigma(t))x(t)$ is less than or equal to the largest real part of the eigenvalues of the matrix $H(\delta)$. The proof of Theorem 3.7 is then straightforward. □

Remark: When $\delta = 2$, Theorem 3.7 reduces to Theorem 3.6 and in this context, Theorem 3.7 is a general sufficient condition for δ-moment stabilizability. The homotopy procedure given in Mariton [27] can be used to numerically solve for $K(1), \ldots, K(N)$. When the dimension of the system and the number of states of the finite-state Markov chain increase, the dimension of the matrix $H(\delta)$ increases, so the above criteria for δ-moment stabilization becomes increasingly complicated. The next result gives a simpler and possibly more useful test for δ-moment stabilization.

Theorem 3.8: Let $\mu(\cdot)$ denote any induced matrix measure, define

$$U(\delta) = \delta \mathrm{diag}\{\mu(A(1) - B(1)K(1)), \ldots, \mu(A(N) - B(N)K(N))\} + Q.$$

If there exist matrices $K(1), \ldots, K(N)$ such that the matrix $U(\delta)$ is Hurwitz stable, then (2.2) is δ-moment stabilizable. In particular, for a one-dimensional system, (2.2) is δ-moment stabilizable if and only if there exists matrices $K(1), \ldots, K(N)$ such that $U(\delta)$ is Hurwitz stable, in this case, $U(\delta) = \delta \mathrm{diag}\{A(1) - B(1)K(1), \ldots, A(N) - B(N)K(N)\} + Q$.

Proof: This can be proved using the δ-moment stability result given in [21] and Coppel's inequality [32]. □

Theorem 3.8 generally depends on the choice of matrix measure. Different choices of the induced matrix measure can give more or less conservative testable conditions for δ-moment stabilization. This was already observed for δ-moment stability in [21]. Selecting an appropriate matrix measure to improve the testable condition is a challenging problem which requires further investigation.

The matrix measure can also be used to obtain criterion for absolute stabilization, keeping in mind that absolute stabilizability is very conservative. If the system is absolutely stabilizable, the properties of the form process are not needed. A preliminary result for absolute stabilizability is given next.

Theorem 3.9:

1) If there exists a matrix measure $\mu(\cdot)$ and matrices $K(1), \ldots, K(N)$ such that $\mu(A(i) - B(i)K(i)) < 0$, then (2.2) is absolutely stabilizable.
2) If (2.2) is absolutely stabilizable, then (2.2) is individual mode stabilizable.
3) For one-dimensional systems, (2.2) is absolutely stabilizable if and only if it is individual mode stabilizable.

Proof: 1) Follows from Coppel's inequality; 2) for any $i \in \underline{N}$, choose an N state Markov chain such that the ith state is absorbing and the rest of the states are transient, the result then follows directly; and 3) follows from 1) and 2). □

IV. Almost-Sure Stabilizability

It is considerably more difficult to obtain general criterion for almost-sure stabilizability than for moment stabilizability. Ezzine and Haddad [30] briefly discussed this problem, and pointed out some of the difficulties. In this section, we study this topic in more detail.

It is well known that controllability implies stabilizability for classical linear systems. However, as discussed earlier, individual mode controllability does not imply mean-square stabilizability. It is surprising that individual mode controllability implies almost-sure stabilizability under fairly general conditions. This result is summarized next.

Theorem 4.1: Assume that $\{\sigma(t)\}$ is a finite-state ergodic Markov chain with invariant measure π. If there exists an $i \in \underline{N}$ such that $(A(i), B(i))$ is controllable and $\pi_i > 0$, then (2.2) is almost surely stabilizable. As a consequence, we conclude that individual mode controllability implies almost-sure stabilizability. □

To prove this, we need the following lemma.

Lemma 4.1: Consider a matrix in the following companion form:

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \vdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \end{pmatrix}$$

with distinct-real eigenvalues $\lambda_1, \ldots, \lambda_n$ with $|\lambda_i - \lambda_j| \geq 1$ $(i \neq j)$, then there exists a constant $M > 0$ and a positive integer k, both independent of $\lambda_1, \ldots, \lambda_n$, and a nonsingular matrix T such that

$$\|T\| \leq M \left(\max_{1 \leq i \leq n} |\lambda_i| \right)^k \quad \|T^{-1}\| \leq M \left(\max_{1 \leq i \leq n} |\lambda_i| \right)^k$$

and

$$T^{-1}AT = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}.$$

Proof of Lemma 4.1: Because A has distinct real eigenvalues, A can be diagonalized over the real field. The transformation matrix is given by

$$T = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \cdots & \lambda_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \lambda_3^{n-1} & \cdots & \lambda_n^{n-1} \end{pmatrix}$$

and $T^{-1}AT = \text{diag}\{\lambda_1, \ldots, \lambda_n\}$. To prove that T satisfies the required condition, we use the 1-norm. Recall that all matrix norms are equivalent over the real field. First

$$T^{-1} = \frac{\text{adj}(T)}{\det(T)} = \frac{\text{adj}(T)}{\prod_{1 \leq i \leq j \leq n} (\lambda_i - \lambda_j)}.$$

With $|\lambda_i - \lambda_j| \geq 1$, $\|T^{-1}\| \leq \|\text{adj}(T)\|$. All entries of T and $\text{adj}(T)$ are polynomials of $\lambda_1, \ldots, \lambda_n$, and there exists an $M > 0$ and a positive integer $k > 0$, both independent of $\lambda_1, \ldots, \lambda_n$, such that

$$\|T\| \leq M \left(\max_{1 \leq i \leq n} |\lambda_i| \right)^k \quad \|T^{-1}\| \leq M \left(\max_{1 \leq i \leq n} |\lambda_i| \right)^k.$$

This completes the proof. □

Now, we are ready to prove Theorem 4.1.

Proof of Theorem 4.1: We first prove the second statement, i.e., the individual mode controllability implies the almost-sure stabilizability. Without loss of generality, we only prove the single input case. For any $j \in \underline{N}$, individual mode controllability assumption implies that $(A(j), B(j))$ is controllable. Therefore, there exists a nonsingular matrix $T_1(j)$ such that

$$T_1(j)A(j)T_1^{-1}(j) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \vdots & 1 \\ x_1(j) & x_2(j) & x_3(j) & \cdots & x_n(j) \end{pmatrix} \stackrel{\text{def}}{=} A_1(j)$$

$$T_1(j)B(j) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \stackrel{\text{def}}{=} B_1(j).$$

Let $\lambda_1, \ldots, \lambda_n$ be negative-real numbers satisfying: $2n \geq |\lambda_i - \lambda_j| \geq 1$ $(i \neq j)$. Choose a matrix $K_1(j)$ such that

$$A_1(j) - B_1(j)K_1(j; \stackrel{\text{def}}{=} \overline{A}(j)$$

has eigenvalues $\lambda_1, \ldots, \lambda_n$ for any $j \in \underline{N}$. Now $\overline{A}(j)$ is in companion form and from Lemma 4.1, there exists $M_1 > 0$, $l > 0$, which are independent of $\lambda_1, \ldots, \lambda_n$ and j, and nonsingular matrices $T_2(j)$ $(j \in \underline{N})$ satisfying

$$\|T_2(j)\| \leq M_1 \left(\max_{1 \leq i \leq n} |\lambda_i| \right)^l$$
$$\|T_2^{-1}(j)\| \leq M_1 \left(\max_{1 \leq i \leq n} |\lambda_i| \right)^l$$

such that

$$T_2^{-1}(j)\overline{A}(j)T_2(j) = \text{diag}\{\lambda_1, \ldots, \lambda_n\} \stackrel{\text{def}}{=} D, \qquad (j \in \underline{N}).$$

Choose the feedback control $u(t) = -K(\sigma(t))x(t)$, where

$$K(j) = K_1(j)T_1(j), \quad T(j) = T_1^{-1}(j)T_2(j), \qquad (j \in \underline{N}).$$

Then the closed-loop system becomes

$$\dot{x}(t) = T(\sigma(t))DT^{-1}(\sigma(t))x(t). \tag{4.1}$$

From the choice of $T_1(j)$ and $T_2(j)$, there exists an $M_2 > 0$ and $m > 0$, both independent of $\lambda_1, \ldots, \lambda_n$ and j, such that

$$\|T(j)\| \leq M_2 \left(\max_{1 \leq i \leq n} |\lambda_i| \right)^m$$
$$\|T^{-1}(j)\| \leq M_2 \left(\max_{1 \leq i \leq n} |\lambda_i| \right)^m.$$

With $\lambda_j < 0$, let $\lambda = \max_{1 \leq i \leq n} \lambda_i$, then there exists an $M_3 > 0$, independent of $\lambda_1, \ldots, \lambda_n$ and j, such that

$$\|e^{Dt}\| \leq M_3 e^{\lambda t}, \qquad (t \geq 0).$$

From the sojourn time description of a finite-state Markov chain, (4.1) is almost surely stable if and only if the state transition matrix

$$\Phi(t, 0) = e^{\tilde{A}(r_k)(t-t_k)} e^{\tilde{A}(r_{k-1})\tau_{k-1}} \cdots e^{\tilde{A}(r_0)\tau_0} \stackrel{t \to \infty}{\longrightarrow} 0$$

almost surely, where $\tilde{A}(j) = T(j)DT^{-1}(j)$ $(j \in \underline{N})$. A straightforward computation yields

$$\begin{aligned}
\|\Phi(t, 0)\| &= \|T(r_k)e^{D(t-t_k)}T^{-1}(r_k)T(r_{k-1}) \\
&\quad \times e^{D\tau_{k-1}}T^{-1}(r_{k-1})\cdots T(r_0)e^{D\tau_0}T^{-1}(r_0)\| \\
&\le \|T(r_k)\|\|e^{D(t-t_k)}\|\|T^{-1}(r_k)\|\|T(r_{k-1})\| \\
&\quad \times \|e^{D\tau_{k-1}}\|\cdots\|T(r_0)\|\|e^{D\tau_0}\|\|T^{-1}(r_0)\| \\
&\le \left[M_2\left(\max_{1\le i\le n}\lambda_i\right)^m\right]^{2(k+1)} \\
&\quad \times M_3^{k+1}e^{\lambda(\tau_k+\tau_{k-1}+\cdots+\tau_0)} \\
&\triangleq \left[Me^{\lambda((\tau_k+\tau_{k-1}+\cdots+\tau_0)/(k+1))}\right]^{k+1} \qquad (4.2)
\end{aligned}$$

where $M = (M_2 \max_{1\le i\le n}|\lambda_i|^l)^2M_3$. $\{\sigma(t)\}$ is a finite-state ergodic Markov chain and from the Law of Large Numbers, there exists a nonrandom constant $a > 0$, the average sojourn time, such that

$$\lim_{k\to\infty}\frac{\tau_k+\tau_{k-1}+\cdots+\tau_0}{k+1} = a \qquad \text{a.s.}$$

Hence

$$\begin{aligned}
&\lim_{k\to\infty} Me^{\lambda((\tau_k+\tau_{k-1}+\cdots+\tau_0)/(k+1))} \\
&\qquad = Me^{\lambda a} \le [M_2(|\lambda\| + 2n)^l]^2M_3e^{\lambda a} \overset{\lambda\to-\infty}{\longrightarrow} 0.
\end{aligned}$$

Thus, we can choose $|\lambda|$ sufficiently large so that $Me^{\lambda a} < 1$. Then, from (4.2)

$$\lim_{t\to\infty}\Phi(t, 0) = 0, \qquad \text{a.s.}$$

i.e., (4.1) is almost surely stable. Therefore, (2.2) is almost surely stabilizable.

Next, we prove the first statement (i.e., the general case). Without loss of generality, we assume that $(A(1), B(1))$ is controllable and $\pi_1 > 0$. We choose $K(2) = K(3) = \cdots = K(N) = 0$, and choose $K(1)$ and λ as in the first half of our previous proof. Then, there exists an $M > 0$ and α, both independent of λ, such that

$$\begin{aligned}
\left\|e^{A(i)t}\right\| &\le Me^{\alpha t}, \qquad (i \ne 1) \\
\left\|e^{(A(1)-B(1)K(1))t}\right\| &\le p(\lambda)e^{\lambda t} \qquad \forall\, t \ge 0
\end{aligned}$$

where $p(\lambda)$ is a polynomial with degree independent of λ. Let γ_k^1 denote the time occupied by state 1 during the time interval $(0, t_k)$ and let γ_k^2 denote the time occupied by the states $2, 3, \ldots, N$ during the interval $(0, t_k)$. From the ergodicity of $\{\sigma(t)\}$

$$\lim_{k\to\infty}\frac{\gamma_k^1}{t_k} = \pi_1 \quad \lim_{k\to\infty}\frac{\gamma_k^2}{t_k} = 1 - \pi_1.$$

As in the first half of the proof, we obtain

$$\|\Phi(t_k, 0)\| \le \left[(Mp(\lambda))^{(k+1)/t_k}e^{\lambda\gamma_k^1/t_k+\alpha\gamma_k^2/t_k}\right]^{t_k}$$

and the term inside $[\cdots]$ has the limit

$$(Mp(\lambda))^{1/a}e^{\pi_1\lambda+(1-\pi_1)\alpha} \longrightarrow 0(\lambda \to -\infty).$$

Therefore, we can conclude that the system is almost surely stabilizable. This completes the proof of Theorem 4.1. □

Remark: It is possible to relax the ergodicity assumption on the process $\{\sigma(t)\}$. In fact, what is required is that the average sojourn time of the process $\{\sigma(t)\}$ is positive for the mode in which the system is controllable.

One may wonder if in Theorem 4.1 individual mode controllability can be relaxed to stabilizability of the individual modes? The answer to this question is no. The following example shows that individual mode stabilizability of a jump linear control system does not imply almost-sure stabilizability.

Example 4.1: (*Individual mode stabilizability does not guarantee δ-moment stabilizability and almost-sure stabilizability*). Let

$$\begin{aligned}
A(1) &= \begin{pmatrix} -a & 1 \\ 0 & -a \end{pmatrix} \quad A(2) = \begin{pmatrix} -a & 0 \\ 1 & -a \end{pmatrix} \\
Q &= \begin{pmatrix} -q & q \\ q & -q \end{pmatrix} \quad B(1) = B(2) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{aligned}$$

where $a > 0$ and $q > 0$ satisfying $1 - a/q \ge 1/2$. The system (2.2) is individual mode stabilizable, however, from Fang [21], the top Lyapunov exponent for this system is positive, hence, (2.2) is almost surely unstable for any control.

Example 4.2: (*Almost-sure stabilizability does not imply individual mode stabilizability*). Let $A(1) = 1, A(2) = 2, B(1) = 0$, $B(2) = 1$ and $Q = \left(\begin{smallmatrix} -1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$. Obviously, $(A(1), B(1))$ is not stabilizable. However, the system is almost surely stabilizable. The invariant measure of the form process $\sigma(t)$ is $\pi = \{1/2, 1/2\}$. A negative feedback control law with $K(1) = 0$ and $K(2) = 10$, almost surely stabilizes the system.

If we choose $A(2) = -10$ and $B(2) = 0$ in example (4.2), then the system is still almost surely stabilizable, and none of individual modes are controllable.

The matrix measure can also be used to derive testable conditions for almost-sure stabilization, some results in this direction are presented next.

Theorem 4.2: Let $\{\sigma(t)\}$ be a finite-state ergodic Markov chain with invariant measure $\pi = \{\pi_1, \pi_2, \ldots, \pi_N\}$. For any matrix measure $\mu(\cdot)$, if there exists matrices $K(1), K(2), \ldots, K(N)$ such that

$$\begin{aligned}
&\pi_1\mu(A(1) - B(1)K(1)) + \pi_2\mu(A(2) - B(2)K(2)) + \cdots \\
&\qquad + \pi_N\mu(A(N) - B(N)K(N)) < 0
\end{aligned}$$

then (2.2) is almost surely stabilizable. Moreover, for one-dimensional systems, the above condition is also necessary.

Proof: Follows from Coppel's inequality and the ergodicity of $\{\sigma(t)\}$. □

By specifying the matrix measure $\mu(\cdot)$ in Theorem 4.2, we can obtain many easy-to-use results for almost-sure stabilization. For example, by using 1-norm, 2-norm (Euclidean norm), and ∞-norm, we can obtain the following.

Corollary 4.1: Suppose that $\{\sigma(t)\}$ is a finite-state ergodic Markov chain with invariant measure π, let $\overline{A}(i) = A(i) - B(i)K(i)$ $(i \in \underline{N})$. The system (2.2) is almost surely stabilizable if there exists matrices $K(1), K(2), \ldots, K(N)$ such that one of the following conditions hold.

1) There exists a positive–definite matrix P such that

$$\sum_{i=1}^{N} \pi_i \lambda_{\max} \left[P\overline{A}(i)P^{-1} + \overline{A}(i)^T\right] < 0.$$

2) There exists positive numbers $r_1, r_2, \ldots, r_N$ such that

$$\sum_{p=1}^{N} \pi_p \max_i \left\{ \overline{a}_{ii}(p) + \sum_{j\neq i} \frac{r_j}{r_i} |\overline{a}_{ij}(p)| \right\} < 0,$$

or

$$\sum_{p=1}^{N} \pi_p \max_j \left\{ \overline{a}_{jj}(p) + \sum_{i\neq j} \frac{r_i}{r_j} |\overline{a}_{ij}(p)| \right\} < 0$$

where $\overline{A}(p) = (\overline{a}_{ij}(p))$.

3)

$$\sum_{p=1}^{N} \pi_p \max_i \left\{ \overline{a}_{ii}(p) + \sum_{j\neq i} |\overline{a}_{ij}(p)| \right\} < 0$$

or

$$\sum_{p=1}^{N} \pi_p \max_j \left\{ \overline{a}_{jj}(p) + \sum_{i\neq j} |\overline{a}_{ij}(p)| \right\} < 0.$$

4)

$$\sum_{i=1}^{N} \pi_i \lambda_{\max} \left[\overline{A}(i) + \overline{A}(i)^T\right] < 0.$$

□

Remarks: Conditions 3) and 4) are special cases of 2) and 1), respectively, and may yield conservative results. As mentioned previously, a similarity transformation is usually required before Corollary 4.1 can be applied.

In order to use 2), the positive numbers $r_1, r_2, \ldots, r_N$ have to be chosen appropriately. Using the following fact from M-matrix theory, a necessary condition for 2) can be obtained: If $A = (a_{ij})$ satisfies $a_{ij} \le 0$ $(i \neq j)$, then there exists positive numbers $r_1, r_2, \ldots, r_n$ such that $a_{ii}r_i > \sum_{j\neq i} r_j|a_{ij}|$ $(i = 1, 2, \ldots, n)$ if and only if A is a Hurwitz matrix or equivalently, all principal minors of A are positive. Let $U = (u_{ij})_{n\times n}$, where

$$u_{ii} = \sum_{p=1}^{N} \pi_p \overline{a}_{ii}(p), \quad u_{ij} = \sum_{p=1}^{N} \pi_p |\overline{a}_{ij}(p)| \qquad (j \neq i).$$

Then, if 2) is satisfied, U is a Hurwitz matrix and all principal minors of $-U$ are positive. From this, to apply 2), it is only necessary to determine if U is Hurwitz. If not, then 2) can not be satisfied. We conjecture that the stability of U is also a sufficient condition for almost-sure stabilizability.

It was shown in [21] that in the parameter space of the system, the domain for δ-moment stability monotonically increases and converges, roughly speaking, to the domain of almost-sure stability as $\delta > 0$ decreases to zero. This implies that almost-sure stability is equivalent to δ-moment stability for sufficiently small $\delta > 0$. From this, we can also say that almost-sure stabilizability is equivalent to δ-moment stabilizability for sufficiently small $\delta > 0$, that is, system (2.2) is almost surely stabilizable if and only if there exists a $\delta > 0$ such that (2.2) δ-moment stabilizable. Thus, almost-sure stabilizability can be studied using δ-moment stabilizability. From this idea, the following general sufficient condition for almost-sure stabilizability is obtained.

Theorem 4.3: Let $\overline{A}(i) = A(i) - B(i)K(i)$ $(i \in \underline{N})$. If there exists matrices $K(1), K(2), \ldots, K(N)$ and positive–definite matrices $P(1), P(2), \ldots, P(N)$ such that for any $i \in \underline{N}$

$$\max_{\|x\|_2=1} \left(\frac{x^T\left[P(i)\overline{A}(i) + \overline{A}^T(i)P(i)\right]x}{x^TP(i)x} + \sum_{j\neq i} q_{ij} \log\left(\frac{x^TP(j)x}{x^TP(i)x}\right)\right) < 0 \quad (4.3)$$

then there exists a $\delta > 0$ such that (2.2) is δ-moment stabilizable, hence it is also almost surely stabilizable.

Proof: This proof is similar to the proof of the almost-sure stability results given in [21]. The Lyapunov function $V(x, \sigma(t)) = (x^TP(\sigma(t))x)^{\delta/2}$ is used and $\delta > 0$ is chosen to be sufficiently small. □

Because this result does not require that the form process is ergodic, Theorem 4.3 is likely to have more applications in practice. The following result shows that Theorem 4.3 is very general sufficient condition for almost-sure stabilizability.

Corollary 4.2:

1) If (2.2) is second moment stabilizable, then there exists matrices $K(1), \ldots, K(N)$ and positive–definite matrices $P(1), \ldots, P(N)$ such that (4.3) is satisfied.
2) For a one-dimensional system, (2.2) is almost surely stabilizable if and only if there exists $K(1), \ldots, K(N)$ and positive numbers $P(1), \ldots, P(N)$ such that (4.3) holds.
3) If there exists matrices $K(1), \ldots, K(N)$ and positive–definite matrices $P(1), \ldots, P(N)$ such that

$$\lambda_{\max}\left\{\left[P(i)\overline{A}(i) + \overline{A}^T(i)P(i)\right]P^{-1}(i)\right\} + \sum_{j\neq i} q_{ij} \log[P(j)P^{-1}(i)] < 0, \qquad (i \in \underline{N})$$

then (2.2) is almost surely stabilizable with feedback control $u(t) = -K(\sigma(t))x(t)$.

Proof: Condition 1) can be proved by using the second moment stabilizability result, 2) can be proved by calculating the explicit solution, and 3) follows directly from (4.3). □

Remarks:

1) The necessary and sufficient condition 2) in Corollary 4.2 for one dimensional systems can be used to obtain some sufficient conditions for almost-sure stabilization for higher dimensional systems. The idea is to use Coppel's inequality to reduce a higher dimensional system to a one dimensional system.
2) Theorem 4.2 can be applied only if the form process is ergodic, condition 2) in Corollary 4.2 may provide a more general sufficient conditions for almost-sure stabilizability. For one dimensional systems using 2) of Corollary 4.2, (2.2) is almost surely stabilizable if and only if

there exists matrices $K(1), \ldots, K(N)$ such that the following series of inequalities hold: ($A \leq_e B$ or $A <_e B$ are elementwise inequalities for the matrices A and B)

$$\exists P(i) > 0, \quad 2\overline{A}(i) + \sum_{j \neq i} q_{ij} \log \frac{P(j)}{P(i)} < 0, \quad (i \in \underline{N})$$

$$\Longleftrightarrow \exists P(i) > 0, \quad 2\overline{A}(i) + \sum_{j=1}^{N} q_{ij} \log P(j) < 0, \quad (i \in \underline{N})$$

$$\Longleftrightarrow \exists y \in R^N, \quad y > 0, \quad \begin{pmatrix} \overline{A}(1) \\ \overline{A}(2) \\ \vdots \\ \overline{A}(N) \end{pmatrix} + Q \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix} <_e 0.$$

Theorem 4.4 formalizes this result.

Theorem 4.4: Let $\mu(\cdot)$ denote any induced matrix measure, and let $a = (\mu(A(1) - B(1)K(1)), \ldots, \mu(A(N) - B(N)K(N)))^T$. If there exists matrices $K(1), \ldots, K(N)$ such that the inequality $a + Qy <_e 0$ has a solution $y \in R^N$, then (2.2) is almost surely stabilizable. Moreover, the solvability of the inequality $a + Qy <_e 0$ is also a necessary condition for almost-sure stabilizability for one-dimensional systems.

Proof: Let $\mu(\cdot)$ be the matrix measure induced by the vector norm $\|\cdot\|$ and let $x(t)$ denote the sample solution of the closed-loop system $\dot{x}(t) = [A(\sigma(t)) - B(\sigma(t))K(\sigma(t))]x(t)$. From Coppel's inequality

$$\|x(t)\| \leq \|x_0\| \exp\left[\int_0^t \mu\left(\overline{A}(\sigma(\tau))\right) d\tau\right]. \qquad (4.4)$$

Consider the system $\dot{z}(t) = \mu[\overline{A}(\sigma(t))]z(t)$ with initial condition $z(0) = \|x_0\|$. Then, the sample solution $z(t)$ is given on the right-hand side of (4.4) and it follows that if $\dot{z}(t) = \mu[\overline{A}(\sigma(t))]z(t)$ is almost surely stable, then from (4.4), (2.2) is almost surely stabilizable with the feedback control $u(t) = -K(\sigma(t))x(t)$. Using the result for one dimensional systems completes the proof. □

As stated earlier, by specifying the matrix measure useful easy-to-use criteria for almost-sure stabilizability can be obtained. Next, we want to show that Theorem 4.4 is more general than Theorem 4.2. In fact, in Fang [21], we showed that if Q and π are the infinitesimal generator and invariant measure, respectively, of a finite-state ergodic Markov chain, then for any vector a, the inequality $a + Qy <_e 0$ has a solution y if and only if $\pi a < 0$. Suppose that $\{\sigma(t)\}$ is a finite-state ergodic Markov chain, from the above fact it follows that Theorem 4.2 and Theorem 4.4 are equivalent. However, when the form process $\{\sigma(t)\}$ is not ergodic, then Theorem 4.2 can not be used, however, Theorem 4.4 can still be applied. This is illustrated in the next example.

Example 4.3: Let $A(1) = a_1$ and $A(2) = a_2$ denote two real numbers and $B(1) = B(2) = 0$. Assume that the form process $\{\sigma(t)\}$ is a two state Markov chain with infinitesimal generator $Q = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. It is obvious that the form process is not ergodic and Theorem 4.2 can not be used. However, from Theorem 4.4, (2.2) is almost surely stabilizable if and only if $a + Qy = a <_e 0$, i.e., $a_1 < 0$ and $a_2 < 0$. From Q, we see that the only uncertainty about the form process is the initial probability distribution.

V. Illustrative Examples

In this section, some examples are given to show how the criteria developed in this paper can be used to study stochastic stabilizability. We first begin with an example motivated by the study of dynamic reliability of multiplexed control systems [17].

Example 5.1: Let

$$A(1) = \begin{pmatrix} 0 & 0 & 0 \\ 1.5 & 0 & 1.5 \\ 0 & 0 & 0 \end{pmatrix} \quad B(1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$K(1) = \begin{pmatrix} k_1 & k_2 & 0 \\ 0 & k_2 & k_1 \end{pmatrix}$$

$$A(2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1.5 \\ 0 & 0 & 0 \end{pmatrix} \quad B(2) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$K(2) = \begin{pmatrix} k_1 & 0 & 0 \\ 0 & k_2 & k_1 \end{pmatrix}$$

$$A(3) = \begin{pmatrix} 0 & 0 & 0 \\ 1.5 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad B(3) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$K(3) = \begin{pmatrix} k_1 & k_2 & 0 \\ 0 & 0 & k_1 \end{pmatrix}$$

$$A(4) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad B(4) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$K(4) = \begin{pmatrix} k_1 & 0 & 0 \\ 0 & 0 & k_1 \end{pmatrix}.$$

This models a first order system with two controllers (incorporating the redundancy principle for reliability) (see [17] for details). The first mode (state 1) corresponds to the case where both controllers are good, and the second and third modes (the states 2 and 3) correspond to the case where one of the controllers fails, and the fourth mode (the state 4) corresponds to the case where both controllers fail. We assume that whenever a controller fails, it will be repaired. Suppose that the failure rate is λ and the repair rate is μ, and the failure process and the repair process are both exponentially distributed. Then the form process is a finite-state Markov chain with infinitesimal generator

$$Q = \begin{pmatrix} -2\lambda & \lambda & \lambda & 0 \\ \mu & -(\lambda+\mu) & 0 & \lambda \\ \mu & 0 & -(\lambda+\mu) & \lambda \\ 0 & \mu & \mu & -2\mu \end{pmatrix}.$$

In [17], Ladde and Šiljak developed a sufficient condition for second moment (mean-square) stabilizability and used this to show that when $\lambda = 0.4$ and $\mu = 0.55$, $k_1 = 2.85$ and $k_2 = 0.33$, the controller $u(t) = -K(\sigma(t))x(t)$ stabilizes the jump linear system (2.2). In this approach, an appropriate choice of positive–definite matrices should be sought, which is very difficult. However, using Theorem 3.8, we can easily check that in this case, the eigenvalues of the matrix H in Theorem 3.8 have negative real parts and H is Hurwitz. It is also easy to show that for the failure rate $\lambda = 0.4$ and the repair rate $\mu = 0.55$, any controller with the parameters k_1 and k_2 satisfying $0.1 \le k_1 \le 5$ and $0.1 \le k_2 \le 5$ can stabilize the second moment of the jump linear system (2.2). Similarly, using the controller with $k_1 = 4$ and $k_2 = 0.55$, for any failure rate $0.4 \le \lambda \le 0.6$ and repair rate $0.4 \le \mu \le 0.6$, the second moment of system (2.2) can be stabilized by this controller. One important fact is that even if the failure rate is greater than the repair rate, this controller can still stabilize the second moment of the system, i.e., the multiplexed control system is reliable. This result is not readily apparent from [17]. From additional computations, whenever the repair rate is greater than the failure rate, this controller can stabilize the second moment of the system.

Example 5.2: In this example, we study the δ-moment stabilization problem for general $\delta > 0$. Consider the one-dimensional jump linear system (2.2) with

$$A(1) = a_1 \quad B(1) = b_1 \quad A(2) = a_2$$
$$B(2) = b_2 \quad Q = \begin{pmatrix} -q & q \\ q & -q \end{pmatrix}.$$

From Theorem 3.10, (2.2) is δ-moment stabilizable if and only if there exists k_1 and k_2 such that

$$\begin{aligned} \delta(a_1 - b_1 k_1) &< q \\ \delta(a_2 - b_2 k_2) &< q \\ \frac{q}{q - \delta(a_1 - b_1 k_1)} \cdot \frac{q}{q - \delta(a_2 - b_2 k_2)} &< 1. \end{aligned} \tag{5.1}$$

1) If $b_1 \ne 0$ and $b_2 \ne 0$, i.e., the system is individual mode controllable, then with k_1 and k_2 such that $a_1 - b_1 k_1 < 0$ and $a_2 - b_2 k_2 < 0$, (5.1) is satisfied, hence (2.2) is δ-moment stabilized by such a controller.

2) 2) If $b_1 \ne 0$ and $b_2 = 0$, then (2.2) is δ-moment stabilizable if and only if $a_2 < q/\delta$. Necessity follows from the second inequality in (5.1). Suppose that $a_2 < q/\delta$, choosing k_1 such that

$$b_1 k_1 > \frac{q(a_1 + a_2) - \delta a_1 a_2}{q - \delta a_2}$$

we can easily verify that (5.1) is satisfied for any k_2, hence, (2.2) is stabilized by such controller.

3) If $b_1 = 0$ and $b_2 \ne 0$, then (2.2) is δ-moment stabilizable if and only if $a_1 < q/\delta$.

4) If $b_1 = b_2 = 0$, then (2.2) is δ-moment stabilizable if and only if

$$\begin{aligned} \delta a_1 &< q \\ \delta a_2 &< q \\ \frac{q}{q - \delta a_1} \cdot \frac{q}{q - \delta a_2} &< 1. \end{aligned}$$

The domain of (a_1, a_2) for which (2.2) is δ-moment stabilizable is illustrated in [18].

Example 5.3: Let

$$A(1) = \begin{pmatrix} 0 & 1 \\ -4 & 10 \end{pmatrix} \quad A(2) = \begin{pmatrix} 0 & 0 \\ -100 & 27 \end{pmatrix}$$
$$B(1) = B(2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad Q = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

The system (2.2) with this data is individual mode controllable and from Theorem 4.1, (2.2) is almost surely stabilizable. We want to find the feedback matrices $K(1)$ and $K(2)$ such that the control $u(t) = -K(\sigma(t))x(t)$ almost surely stabilizes (2.2). Theorem 4.2 is used to solve this problem. First notice that the invariant measure for the form process is $\pi = (1/2, 1/2)$. Choose $K(2) = (-100, 27)$, then $\overline{A}(2) = A(2) - B(2)K(2) = 0$, and for any matrix measure $\mu(\cdot)$, $\mu(\overline{A}(2)) = 0$. For the first mode, we can choose a $K(1)$ such that the eigenvalues of $A(1) - B(1)K(1)$ can be assigned, for example, to -1 and -3. This can be achieved by setting $K(1) = (-2, 13)$, then $\overline{A}(1) = A(1) - B(1)K(1) = \left(\begin{smallmatrix} 0 & 1 \\ -2 & -3 \end{smallmatrix}\right)$. Then,

$$TAT^{-1} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}, \qquad \text{where } T = \begin{pmatrix} 1 & 1 \\ -1 & -3 \end{pmatrix}^{-1}.$$

Define the vector norm $\|x\|_T = \|Tx\|_2$, the induced matrix measure is given by $\mu_T(A) = \mu_2(TAT^{-1})$. From this, we can easily compute

$$\mu_T\left(\overline{A}(1)\right) = \mu_2\left(\begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix}\right) = -1,$$
$$\mu_T\left(\overline{A}(2)\right) = \mu_T(0) = 0.$$

Then $\pi_1 \mu_T(\overline{A}(1)) + \pi_2 \mu_T(\overline{A}(2)) = (1/2) \times (-1) + (1/2) \times 0 = -(1/2) < 0$ and from Theorem 4.2 the controller $u(t) = -K(\sigma(t))x(t)$ with $K(1) = (-2, 13)$ and $K(2) = (-100, 27)$ almost surely stabilizes the system.

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