Systems & Control Letters 23 (1994) 237–245
North-Holland

A sufficient condition for stability of a polytope of matrices

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Received 27 June 1993
Revised 16 October 1993

Abstract: In this paper, we study the stability of interval dynamical systems. A sufficient condition for the stability of a polytope of matrices, which is shown to be necessary and sufficient for a certain class of matrices, is obtained. The results developed can also be applied to the stability of a positive cone of matrices and sufficient conditions for the stability of interval dynamical systems are obtained. A relationship between real parts of eigenvalues and matrix measures is also presented.

Keywords: Stability; interval dynamical systems; matrix measure; robustness.

1. Introduction

The stability of interval dynamical systems has attracted considerable interest since the publication of Kharitonov's paper [12]. The formulation is simple: suppose that a dynamical system is described by a set of linear differential or difference equations with a corresponding set of parameters. If the parameters are not exactly known, but the boundary of the parameter set is known, the problem is to determine if the stability of the system can be determined from the stability of the system when the parameters are at the boundary values of the admissible set. In his short paper, Kharitonov [12] proved that the stability of dynamical systems whose parameter uncertainty is restricted to a rectangular domain is guaranteed by the stability of properly chosen dynamical systems whose parameters take values at the vertices of the parameter domain. This work stimulated active research in this area, and the results which have been reported are closely related to the robustness properties of dynamical systems. Jury [11] presented a comprehensive survey summarizing the research results, especially for discrete time dynamical systems.

Given a linear state space model of a dynamical system, stability is determined by the eigenvalues of the system matrix. When the parameters in the system matrix are uncertain, we are interested in the stability of the matrix given this uncertainty, i.e., the robust stability of the dynamical system with respect to these parameter variations. The most interesting case is when the uncertainty in each parameter in the system matrix is modeled by an interval, i.e., the lower and upper bounds are known. A natural conjecture would be that the family of system matrices is stable if and only if all the vertex matrices are stable. Bialas [4] claimed that this is true. But it turned out to be false, Barmish and Hollot [3] constructed a counterexample. Similarly, Jiang [10] attempted to prove the above conjecture for the discrete case, which is also not true as pointed out by Soh [18]. Sufficient conditions have been obtained and for special vertices in the parameter domain, it is desirable to obtain necessary and sufficient conditions for robust stability. Heinen [7] used an extension of Gershgorin's circle theorem to obtain a simple sufficient condition for the stability of the family of system matrices, which is too conservative. Argoun [1] also used Gershgorin's theorem to obtain sufficient conditions, which are incorrect [23]. Soh [20] corrected Argoun's approach and obtained a set of sufficient

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SSDI 0167-6911(93)E0141-3

conditions. However, Soh's result depends on the diagonalizability of the arithmetic average of the vertex matrices, and for this reason it may find limited applications. Jiang [9] and Mansour [14, 15] obtained a more general set of sufficient conditions. For a special class of vertices, it is possible to obtain a necessary and sufficient condition for the stability of the matrix family. Mori and Kokame [16] presented a necessary and sufficient condition for a family of matrices whose vertices have norm less than 1. Xu [22] and Liao [13] studied the matrix family whose vertices are M-matrices and obtained necessary and sufficient conditions. Shi and Gao [17] and Soh [19] proved that for a matrix family whose vertices are symmetric, the corresponding interval dynamical systems are stable if and only if the vertex matrices are stable. Wang [21] generalized this result to the case of normal vertex matrices. Barmish et al. [2] presented some counter-examples to plausible conjectures relating to the stability of interval dynamical systems.

In this paper, we will prove a very general sufficient condition for a matrix family such that stability of the vertices guarantees stability of the convex hull and positive cone of the matrix family. This result implies the results in [1, 2, 9, 14, 15, 17, 19, 20]. For a certain class of vertices, the condition becomes necessary and sufficient. From this, a sufficient condition for stability of interval dynamical systems is obtained. We also obtain a formula which illustrates the relationship between the real parts of eigenvalues and the matrix measure.

2. Main results

Consider a convex hull (polytope) of matrices in the set $\mathbb{R}^{n\times n}$ of $n\times n$ matrices described by

$$P=\left\{A \mid A=\sum_{i=1}^{m}\alpha_i A_i,\ \alpha_i\geqslant 0,\ \sum_{i=1}^{m}\alpha_i=1\right\}.$$

We say the matrix A is stable if its eigenvalues have negative real parts. We say that P is S-stable if each matrix in P is stable. As we know, the stability of some interval dynamical systems can be reduced to S-stability [2]. In what follows, we use Re to denote the real part of a complex number and $\lambda(A)$ or $\lambda_i(A)$ to denote the eigenvalue of matrix A.

To state the sufficient conditions some definitions are required, which are given next.

Let $|x|$ denote a vector norm of x on $\mathbb{C}^n$, and let $\|A\|$ be the induced matrix norm of A given the vector norm $|\cdot|$. $\mu(A)$ is the matrix measure of A defined as

$$\mu(A)\triangleq\lim_{\theta\downarrow 0^+}(\|I+\theta A\|-1)/\theta,$$

where I is identity matrix. The matrix measure gives an upper bound for the magnitude of the solution of a differential equation $\dot{x}(t)=A(t)x(t)$, i.e., the following well-known Coppel inequality [5]:

$$\|x(t)\|\leqslant\|x(t_0)\|\exp\left(\int_{t_0}^{t}\mu(A(\tau))\,\mathrm{d}\tau\right),$$

which renders it suitable for stability investigations.

Lemma 1 (Desoer [5]). *$\mu(A)$ is well defined for any norm and has the following properties:*

(a) *μ is convex on $\mathbb{C}^{n\times n}$, i.e., for any $\alpha_j\geqslant 0$ $(1\leqslant j\leqslant k)$, and $\sum_{j=1}^{k}\alpha_j=1$, and any matrices A_j $(1\leqslant j\leqslant k)$, we have*

$$\mu\left(\sum_{j=1}^{k}\alpha_j A_j\right)\leqslant\sum_{j=1}^{k}\alpha_j\mu(A_j).$$

(b) *For any norm, and any A, we have*

$$-\|A\|\leqslant-\mu(-A)\leqslant\operatorname{Re}\lambda(A)\leqslant\mu(A)\leqslant\|A\|.$$

(c) *For the* 1-*norm* $|x|_1 = \sum_{i=1}^{n} |x_i|$, *the induced matrix measure* μ_1 *is given by*

$$\mu_1(A) = \max_j \left[\mathrm{Re}(a_{jj}) + \sum_{i \neq j} |a_{ij}| \right];$$

for the 2-*norm* $|x|_2 = (\sum_{i=1}^{n} |x_i|^2)^{1/2}$, *the induced matrix measure* μ_2 *is given by*

$$\mu_2(A) = \max_i [\lambda_i(A + A^*)/2];$$

for the ∞-*norm* $|x|_\infty = \max_{1 \leqslant i \leqslant n} |x_i|$, *the induced matrix measure is given by*

$$\mu_\infty(A) = \max_i \left[\mathrm{Re}(a_{ii}) + \sum_{j \neq i} |a_{ij}| \right].$$

(d) *For any nonsingular matrix* T *and any vector norm* $\|\cdot\|$ *with the induced matrix measure* μ, $\|Tx\|$ *defines another vector norm and its induced matrix measure* μ_T *is given by*

$$\mu_T(A) = \mu(TAT^{-1}).$$

Theorem 1. *If there exists a norm such that* $\mu(A_j) < 0$ $(1 \leqslant j \leqslant m)$, *then* P *is S-stable.*

Proof. Let A be any matrix in P, then there exist $\alpha_j \geqslant 0$ and $\sum_{j=1}^{m} \alpha_j = 1$, so that $A = \sum_{j=1}^{m} \alpha_j A_j$. From the convexity of μ and the assumption that $A \in P$, we obtain

$$\mu(A) \leqslant \sum_{j=1}^{m} \alpha_j \mu(A_j) < 0.$$

Then using Lemma 1(b), we have $\mathrm{Re}\,\lambda_i(A) \leqslant \mu(A) < 0$, which means that A is stable. Therefore, P is S-stable. This completes the proof. □

In fact, we can generalize this result to the case where A is in a positive cone. Let

$$Q = \left\{ A \mid A = \sum_{j=1}^{m} \alpha_j A_j \mid \alpha_j \geqslant 0,\ A \neq 0 \right\}$$

be a cone of matrices, which is referred to as a positive cone. We say that Q is S-stable if each nonzero matrix in Q is stable. Then we have the following theorem.

Theorem 2. *If there exists a matrix measure* μ *such that* $\mu(A_j) < 0$ $(1 \leqslant j \leqslant m)$, *then* Q *is S-stable.*

Proof. Using the following two properties of the matrix measure [5]:
(i) $\mu(A + B) \leqslant \mu(A) + \mu(B)$ for any two matrices A and B,
(ii) $\mu(cA) = c\mu(A)$ for any A and $c \geqslant 0$,
the proof of Theorem 2 follows from the proof of Theorem 1. □

Although the conditions of Theorems 1 and 2 are only sufficient, for a certain class of vertices the condition is also necessary. Let $*$ denote the complex conjugate transpose operation, and let U be a set of matrices. We say that U is $*$-closed if $A^* \in U$ for any $A \in U$. This leads to the following theorem.

Theorem 3. *Let* V *be the set of vertex matrices for a polytope* P, *and let* V *be* $*$-*closed. Then* P *is S-stable if and only if there exists a matrix measure* μ *such that* $\mu(A) < 0$ *for any* $A \in V$.

Proof. The sufficiency follows from Theorem 1. To prove the necessity, we choose the vector 2-norm. For any $A \in V$, we have $A^* \in V$ since V is $*$-closed. From the S-stability of P, we obtain that $(A + A^*)/2$ is stable, i.e., $\max_i \lambda_i(A + A^*)/2 < 0$. From Lemma 1(c), we obtain $\mu_2(A) < 0$. This completes the proof. □

Next, we show that the previously available results can be obtained as corollaries of the main results of this paper. In the following, we always use V to denote the set of vertex matrices.

Corollary 1 (Wang [21]). *If the vertex matrices are normal, then P is S-stable if and only if V is S-stable.*

Proof (*Sufficiency*). For any $A \in V$, since A is normal, from Theorem 2.5.4 of [8, p. 101], we know that A can be diagonalized using a unitary transformation, i.e., there exists a unitary matrix U such that $A = UDU^{-1}$. Using the 2-norm, which is invariant under a unitary transformation, the induced matrix measure is also invariant, and we have

$$\mu_2(A) = \mu_2(UDU^{-1}) = \mu_2(D) = \max_i \operatorname{Re} \lambda_i(D) = \max_i \operatorname{Re} \lambda_i(A).$$

Hence, if A is stable, then $\mu_2(A) < 0$ for any $A \in V$, and from Theorem 1 we conclude that P is S-stable.
The proof of necessity is trivial. This completes the proof. □

Corollary 2 (Shi and Gao [17] and Soh [19]). *If the vertex matrices are symmetric, then P is S-stable if and only if V is S-stable.*

Proof. This is a direct application of Theorem 3. □

Corollary 3 (Jiang [9] and Mansour [14, 15]). *If for any $A \in V$, $(A + A^{\mathrm{T}})/2$ is stable, then P is S-stable.*

Proof. We choose the vector 2-norm; the proof follows from Theorem 1. □

A more general case than Corollary 3 is the following result.

Corollary 4. *If there exists a positive matrix S such that $SA + A^{\mathrm{T}}S$ is stable for any $A \in V$, then P is S-stable.*

Proof. If S is positive definite, let $S = H^2$ where H is also positive definite, then, when we use the norm $\|x\| = \|Hx\|_2$, the induced matrix measure μ_H is given by

$$\mu_H(A) = \mu_2(HAH^{-1}) = \max_i \operatorname{Re} \lambda_i\left(\frac{HAH^{-1} + H^{-\mathrm{T}}A^{\mathrm{T}}H^{\mathrm{T}}}{2}\right)$$
$$= \max_i \operatorname{Re} \lambda_i\left(H^{-1}\frac{SA + A^{\mathrm{T}}S}{2}H^{-1}\right) \leqslant \frac{\max_i \lambda_i(SA + A^{\mathrm{T}}S)}{2\alpha(S,A)},$$

where $\alpha(S,A) = \max_i \lambda_i(S)$ if $SA + A^{\mathrm{T}}S$ is negative semidefinite and $\alpha(S,A) = \min_i \lambda_i(S)$ if $SA + A^{\mathrm{T}}S$ is positive semidefinite. If $SA + A^{\mathrm{T}}S$ is stable for any $A \in V$, then it is negative definite; hence, $\mu_H(A) < 0$ for any $A \in V$. From Theorem 1 we conclude that P is S-stable. □

Corollary 5 (Argoun [1] and Soh [20]). *Let $P_0 = \sum_{i=1}^{m} A_i/m$, and let there exist a nonsingular T such that TP_0T^{-1} is diagonal. Then, if there exists a matrix measure μ such that $\mu(TA_iT^{-1}) < 0$ $(1 \leqslant i \leqslant m)$, P is S-stable.*

Proof. Given any vector norm $|\cdot|$, with T nonsingular, we choose a new norm $\rho(x) = |Tx|$. The new induced matrix measure is $\mu_\rho(A) = \mu(TAT^{-1})$, from Theorem 1, and the proof is complete. □

Remark 1. Theorem 1 implies that we do not need to use a similarity transformation which diagonalizes P_0. Suppose that P_0 is not diagonalizable, then Soh's criterion fails. For example, let

$$A_1 = A_2 = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix},$$

then $P_0 = A_1$ is not diagonalizable and Soh's result is not applicable. However, an easy computation gives $\mu_2(A_1) = \mu_2(A_2) = -0.5 < 0$, and Theorem 1 can be applied. Theorem 1 can be interpreted as a generalization of Soh's result.

Using the vector 1-norm and the vector ∞-norm, we can obtain Heinen's [7] result, which is considered to be too conservative. Xu's [22] and Liao's [13] results are also related to this type of criteria.

The above criteria can be applied to the stability of interval matrices. Let $L = (l_{ij})$, $U = (u_{ij})$ and $A = (a_{ij})$ be real matrices, and let $L \leqslant A \leqslant U$ hold elementwise. Define $K = \{A \mid L \leqslant A \leqslant U\} = \{(a_{ij}) \mid l_{ij} \leqslant a_{ij} \leqslant u_{ij}\}$, and $V = \{A \mid a_{ij} = l_{ij} \text{ or } a_{ij} = u_{ij}\}$, i.e., V is the set of vertex matrices of K. Then from Theorem 1 we have the following result.

Corollary 6. *K is S-stable if there exists a matrix measure μ such that $\mu(A) < 0$ for any $A \in V$.* □

It may seem that the criteria developed in this paper are conservative because the matrix measure is only an upper bound of the real parts of the eigenvalues of a matrix. The following theorem derives the relationship between the real parts of the eigenvalues and the matrix measure. This shows that if we properly choose the matrix measure, we can obtain criteria for stability which are equivalent to testing the real parts of the eigenvalues for negativity.

Theorem 4. *Let $\mathcal{N}$ be the set of all vector norms on $\mathbb{C}^n$; for any $\rho \in \mathcal{N}$, the corresponding matrix measure is denoted as μ_ρ. Then for any matrix in $\mathbb{C}^{n\times n}$, we have*

$$\max_{1 \leqslant i \leqslant n} \operatorname{Re} \lambda_i(A) = \inf_{\rho \in \mathcal{N}} \mu_\rho(A).$$

Proof. Let J be the Jordan form of A, Then from Jordan's theorem, there exists a nonsingular matrix T such that

$$J = TAT^{-1} = D + U,$$

where D is diagonal, and U is a matrix whose diagonal elements are zeros and off-diagonal elements are the same as J. Let $\Lambda \triangleq \operatorname{diag}\{1, \varepsilon^{-1}, \varepsilon^{-2}, \ldots, \varepsilon^{-(n-1)}\}$ for any positive real ε, then

$$\Lambda J \Lambda^{-1} = D + \varepsilon U.$$

Define a new norm of $\mathbb{C}^n$ by $|x| = |\Lambda T x|_2$. Then the corresponding matrix norm of $I + \theta A$, where I denotes the identity matrix, and θ is any positive number, is

$$\|I + \theta A\| = \max_{|x|=1} |(I + \theta A)x| = \max_{|\Lambda T x|_2 = 1} |\Lambda T(I + \theta A)x|_2.$$

Note that, if we let $z \triangleq \Lambda T x$, then we have $\Lambda T(I + \theta A)x = \Lambda T(I + \theta A)T^{-1}\Lambda^{-1}z = I + \theta(D + \varepsilon U)z$, so the corresponding matrix measure of A is

$$\mu(A) = \lim_{\theta \searrow 0} \frac{\|I + \theta A\| - 1}{\theta} = \lim_{\theta \searrow 0} \frac{\max_{|z|_2=1} |z + \theta(D + \varepsilon U)z|_2 - 1}{\theta}$$

$$\leqslant \lim_{\theta \searrow 0} \frac{\max_{|z|_2=1} (|(I + \theta D)z|_2 + \varepsilon\theta |Uz|_2) - 1}{\theta} = \mu_2(D) + \varepsilon |U|_2 \leqslant \max_i \operatorname{Re} \lambda_i(A) + \varepsilon.$$

This means that for any positive real ε, there exists a norm ρ such that

$$\mu_\rho(A) < \max_{1 \leqslant i \leqslant n} \operatorname{Re} \lambda_i(A) + \varepsilon$$

and the proof of Theorem 4 is complete. □

This theorem suggests that in order to obtain the tightest stability bounds, the norm in Theorem 1 must be chosen properly. One possible choice is the class given by the norms $|x|_T = |Tx|_2$ for any nonsingular matrix T. The induced matrix measure is $\mu_T(A) = \mu_2(TAT^{-1})$. This relates to the representation problem for the matrix A. For the stability of interval polynomials, it is interesting to note that the stability of the system depends on finding a suitable state space realization so that the results of Theorem 1 can be applied. This is an interesting observation and will be investigated later. Next, we state an obvious corollary of Theorem 4.

Corollary 7. *A is stable if and only if there exists a matrix measure μ such that $\mu(A) < 0$.* □

The robustness of the stability property of time-invariant linear systems with parameter variations for unmodeled dynamics is also of special interest. Using the concept of the matrix measure, we can obtain the following result.

Theorem 5. *Suppose that the dynamical system is described by*

$$\dot{x}(t) = (A + \Delta A)x(t),$$

where ΔA represents the unmodeled dynamics, then the system is stable if the unmodeled dynamics satisfies the following conditions:

$$\mu(\Delta A) < -\mu(A),$$

where μ is a certain matrix measure.

Proof. Using the property $\mu(A + B) \leqslant \mu(A) + \mu(B)$, the proof follows directly. □

Wang [21] remarked that his results cannot be used to test the polytope of triangular vertex matrices. However, our results can be applied to this case, which shows that our criteria are really more general results. As an illustration, we present a proof (although it can be proved easily from a different approach).

Corollary 8. *If the vertex matrices can be simultaneously transformed by a similarity transformation to upper (or lower) triangular form, then a necessary and sufficient condition for P to be S-stable is that the vertex matrices are stable.*

Proof. Without loss of generality, we assume that all vertex matrices $A_1, A_2, \ldots, A_m$ are upper triangular. Let $A_k = (a_{ij}^k)$ $(k = 1, 2, \ldots, m)$ and $R = \operatorname{diag}\{r_1, r_2, \ldots, r_n\}$. Let μ denote the matrix measure induced by the vector norm $\|R^{-1}x\|_\infty$. Since

$$R^{-1}A_kR = \begin{pmatrix} a_{11}^k & (r_2/r_1)a_{12}^k & \cdots & (r_n/r_1)a_{1n}^k \\ & a_{22}^k & \cdots & (r_n/r_2)a_{2n}^k \\ & & \ddots & \vdots \\ & & & a_{nn}^k \end{pmatrix}, \quad k = 1, 2, \ldots, m,$$

we obtain from Lemma 1(c) and (d)

$$\mu(A_k) = \mu_\infty(R^{-1}A_kR) = \max_i \left[\operatorname{Re}\left(a_{ii}^k + \sum_{j>i} (r_j/r_i)|a_{ij}^k| \right) \right].$$

Let $r_j = \varepsilon^j$ $(j = 0, 1, \ldots, n-1)$, where ε is a sufficiently small positive number. Define $M = \max_{1 \leqslant k \leqslant m} \max_{i,j} |a_{ij}^k|$. Then we have

$$\mu(A_k) \leqslant \max_i \left[\operatorname{Re}\left(a_{ii}^k + M \sum_{j>i} r_j/r_i \right) \right] \leqslant \max_i [a_{ii}^k] + nM\varepsilon.$$

Thus, if A_k is stable, then we can choose $\varepsilon > 0$ sufficiently small, so that $\mu(A_k) < 0$ for any $k = 1, 2, \ldots, m$. From Theorem 1 we conclude that P is S-stable. The necessity is trivial. □

Corollary 9. *If the vertex matrices commute, then a necessary and sufficient condition for P to be S-stable is that the vertex matrices are stable.*

Proof. We use Theorem 2.3.3 of [8, p. 81]. If a family of matrices commute, then they can be simultaneously transformed to upper triangular form, and Corollary 8 provides the desired result. □

Corollary 10. *If the vertex matrices can be simultaneously transformed to Jordan form, then a necessary and sufficient condition for P to be S-stable is that the vertex matrices are stable.* □

3. Examples

Example 1. Let

$$A_1 = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -12 & -3 \\ 4 & 5 \end{pmatrix}, \quad T = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}.$$

Then

$$TA_1T^{-1} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}, \quad TA_2T^{-1} = \begin{pmatrix} -9 & 2 \\ 0 & -8 \end{pmatrix}.$$

Define the vector norm $|x| = |Tx|_2$. Then the induced matrix measure is $\mu(A) = \mu_2(TAT^{-1})$. Thus, $\mu(A_1) = -1 < 0$ and $\mu(A_2) = -7.382 < 0$ and from Theorem 1 we see that P is S-stable.

Remark 2. Since A_1 is not normal, we cannot use Wang's result [21]. Since $\mu_2(A_1) = 0.081 > 0$, we cannot use Jiang's results [9] or Mansour's [14, 15] results. Since

$$TP_0T^{-1} = T(A_1 + A_2)T^{-1}/2 = \begin{pmatrix} -5 & 1 \\ 0 & -5 \end{pmatrix},$$

which is a Jordan block, P_0 cannot be diagonalized; hence, we cannot use Soh's result [20].

Note that for any $R = \mathrm{diag}\{r_1, r_2\}$, $\mu_R(A) = \mu_\infty(R^{-1}AR)$, which is the induced matrix measure by the vector norm $\|R^{-1}\|_\infty$. Then for any positive numbers r_1 and r_2,

$$\mu_R(A_2) = \mu_\infty(R^{-1}A_2R) = 4(r_1/r_2) + 5 > 0.$$

Heinen's [7] results fail the stability test.

Example 2. Consider the interval dynamical system

$$\dot{x}(t) = Ax(t),$$

where

$$A \in A_I = \begin{pmatrix} [-5, -3] & [1, 2] \\ [4, 5] & [-6, -4] \end{pmatrix}.$$

Since

$$\mu_2\left(\begin{pmatrix} -3 & 2 \\ 7 & -4 \end{pmatrix}\right) = \tfrac{1}{2}(-7 + \sqrt{82}) > 0,$$

the criterion by using the 2-norm matrix measure developed in [9, 14, 15] cannot be used to study the robust stability of A_I.

Choose $R = \text{diag}\{1, 0.5\}$. Then

$$RA_IR^{-1} = \begin{pmatrix} [-5, -3] & [2, 4] \\ [2, 2.5] & [-6, -4] \end{pmatrix}.$$

Let

$$L = \begin{pmatrix} -5 & 2 \\ 2 & -6 \end{pmatrix}, \quad U = \begin{pmatrix} -3 & 4 \\ 2.5 & -4 \end{pmatrix}$$

and

$$V = \{A \mid a_{ij} = l_{ij} u_{ij}\}.$$

$\forall A \in V$, the characteristic polynomial of $A^T + A$ is

$$\lambda^2 - 2(a_{11} + a_{22})\lambda + 4a_{11}a_{22} - (a_{12} + a_{21})^2.$$

Since $a_{11} < 0$, $a_{22} < 0$,

$$\mu_R(A) < 0 \iff A + A^T \text{ stable}$$
$$\iff 4a_{11}a_{22} - (a_{12} + a_{21})^2 > 0.$$

For $A \in V$,

$$4a_{11}a_{22} - (a_{12} + a_{21})^2 \geqslant 4(-3)(-4) - (4 + 2.5) > 0 \iff \mu_R(A) < 0 \iff A_I \text{ is stable.}$$

Thus, our criterion is still applicable.

Example 3. Consider

$$\mathscr{U} = \left\{\Delta A \mid \Delta A = \sum_{i=1}^{r} \sigma_i A_i, |\sigma_i| \leqslant \delta_i, i = 1, 2, \ldots, r\right\},$$

where the matrices A_i, which represent the uncertain structure, are given skew matrices, i.e., $A_i^T = -A_i$ $(i = 1, 2, \ldots, r)$. The problem is to find the condition such that $A + \Delta A$ is asymptotically stable for any $\Delta A \in \mathscr{U}$ (we say that $A + \mathscr{U}$ is stable), where A satisfies $A + A^T$ is negative definite. The above formulation can be viewed as the representation of a dissipative system (such as a flexible structure) with energy-conserving perturbation. If we choose the 2-norm, the induced matrix measure is μ_2, which satisfies $\mu_2(A + \Delta A) = \frac{1}{2}\lambda_{\max}(A + A^T) < 0$, so under the above assumptions we conclude that $A + \mathscr{U}$ is stable with convergence rate at least $-\mu_2(A)$.

Example 4. Let

$$A = \begin{pmatrix} 0 & 1 \\ -a_2 & -a_1 \end{pmatrix}.$$

This is the companion (controllability form for a controllable single input system) of a state matrix with the characteristic polynomial $\lambda^2 + a_1\lambda + a_2$. A simple computation yields

$$\mu_2(A) = \tfrac{1}{2}[-a_1 + \sqrt{a_1^2 + (1 - a_2)^2}],$$

and we observe that $\mu_2(A) \geqslant 0$, which is not good for our analysis. The above form is very useful for problems related to robust stabilization and simultaneous stabilizability of an appropriate state feedback control design study for a single input system, but it is not appropriate for stability studies of interval matrices. The study of stability of interval dynamic systems using the criteria developed in this paper requires that a state space realization for these systems has been determined, which forms a future research topic.

4. Conclusion

In this paper, we have obtained a very general sufficient condition for the stability of a polytope of matrices or a positive cone of matrices. For a certain class of vertex matrices, the sufficient condition is also necessary. A sufficient condition for the stability of an interval dynamical system is given and the relationship between the real parts of the eigenvalues of a matrix and its matrix measure is derived. Using the matrix measure, we have also given a result for the stability of time-invariant linear systems with parameter variations representing the unmodeled dynamics.

Acknowledgement

This research was supported in part by the Scientific Research Laboratory at Ford Motor Company, Detroit, Michigan.

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