New robust stability results for linear systems with structured non-linear perturbations

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Abstract: The robust stability of linear systems with unstructured, structured or non-linear perturbations is investigated in a unifying framework. New sufficient conditions in terms of matrix Riccati inequalities are obtained. Some previously known results are generalized.

Keywords: stability, robust stability, Riccati inequality, non-linear perturbation, structured perturbation

NOTATION

$\mathbf{A}^{\mathrm{T}}$	transpose of matrix $\mathbf{A}$
$\Vert\mathbf{A}\Vert_2$	Euclidean norm of matrix $\mathbf{A} = \sigma_{\mathrm{M}}(\mathbf{A})$
$\Vert\mathbf{H}(s)\Vert_\infty$	$\mathbf{H}_\infty$ norm
$\mathbf{I}$	identity matrix with appropriate dimension
j	the imaginary part, i.e. $\sqrt{-1}$
$\mathbf{R}^{1/2} = \sqrt{\mathbf{R}}$	positive definite matrix such that $\mathbf{R} = (\sqrt{\mathbf{R}})^2$
$\mathbf{X} \leqslant \mathbf{Y}$ ($\mathbf{X} < \mathbf{Y}$)	$\mathbf{Y} - \mathbf{X}$ is positive semi-definite (positive definite)
$\lambda_{\mathrm{M}}(\mathbf{X})$	largest eigenvalue of symmetrical matrix $\mathbf{X}$
$\sigma_{\mathrm{M}}(\mathbf{A})$	largest singular value of matrix $\mathbf{A}$

1 INTRODUCTION

Robust stability analysis and designs for dynamical systems with parameter uncertainty have attracted considerable attention since the publication of Kharitonov's celebrated theorem [**1**] on the robust stability of a polynomial with interval parameter uncertainty. Šiljak [**2**] and Mansour [**3**] have presented good surveys on parameter space methods for robust stability analysis of continuous state space models and Jury [**4**] has investigated the generalization of continuous-time robust stability results to discrete-time systems. The books by Mansour *et al.* [**5**] and by Dorato and Yedavalli [**6**] are collections of papers which represent recent developments in this area. There is a great deal of literature on the robust stability of polynomials with uncertain parameters, but comparatively little attention has been paid to the robust stability of state space models. The stability radius concept has been used to study the robustness of stability of state space models with unstructured uncertainty (see reference [**7**] and the references therein). When state space models have structured uncertainty, the stability radius concept may be too conservative and other stability robustness measures may be more suitable. Fang *et al.* [**8–10**] used a matrix measure to study the stability of linear systems with convex perturbations and obtained upper bounds for perturbations with which the system is still stable. A very general approach to solving such problems is Lyapunov's second method. The idea is to use a nominal system to construct a Lyapunov function and then to apply this Lyapunov function to study the stability of the uncertain system. Patel and Toda [**11**] used this approach to study the stability of linear systems with unstructured perturbation and obtained the upper Euclidean bound of the perturbation. Yedavalli [**12**] obtained an upper bound for the interval perturbation for robust stability using this approach. Zhou and Khargonekar [**13**] studied the robust stability of systems with perturbation in the form of a linear combination of a finite number of matrices and improved Yedavalli's result. Yedavalli and Liang [**14**] used a state space transformation before applying Yedavalli's results in reference [**12**] to reduce the conservatism. Yedavalli [**15**] has summarized some recent results in this area. Recently, Fang and Loparo [**16**] have applied Lyapunov's second method and optimization techniques to improve many previously known results and many new results have been obtained.

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In the present paper another Lyapunov function is used for systems with structured (or unstructured) and non-linear perturbations. From this, some new robust stability results are obtained which are expressed in terms of matrix Riccati inequalities.

2 PRELIMINARIES

Matrix $\mathbf{A}$ is said to be stable if all eigenvalues of $\mathbf{A}$ have negative real parts. For a transfer matrix $\mathbf{H}(s)$ the $\mathbf{H}_\infty$ norm is defined as the upper bound of the largest singular value of $\mathbf{H}(j\omega)$ over $[0, +\infty)$, i.e. the upper bound of the largest singular value of the transfer matrix over the imaginary axis (see reference [17]).

The following two lemmas will be used below.

Lemma 1 [17, 18]

Given a scalar $\gamma > 0$, the following statements are equivalent:

1. $\mathbf{A}$ is stable and

$$\|\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\|_\infty < \gamma$$

2. There exists a positive definite matrix $\mathbf{P}$ such that

$$\mathbf{A}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{A} + \gamma^{-2}\mathbf{P}\mathbf{B}\mathbf{B}^{\mathrm{T}}\mathbf{P} + \mathbf{C}^{\mathrm{T}}\mathbf{C} < 0$$

It is also easy to show the following result.

Lemma 2

For any matrices $\mathbf{X}$ and $\mathbf{Y}$ and any positive definite matrix $\mathbf{R}$,

$$\mathbf{X}\mathbf{Y}^{\mathrm{T}} + \mathbf{Y}\mathbf{X}^{\mathrm{T}} \leqslant \mathbf{X}\mathbf{R}^{-1}\mathbf{X}^{\mathrm{T}} + \mathbf{Y}\mathbf{R}\mathbf{Y}^{\mathrm{T}}$$

3 MAIN RESULTS

Consider the following non-linear system:

$$\dot{x}(t) = \mathbf{A}x(t) + \sum_{i=1}^{m} g_i[x(t), t]$$
$$g_i^{\mathrm{T}}[x(t), t]g_i[x(t), t] \leqslant x^{\mathrm{T}}(t)\mathbf{W}_i x(t), \qquad i = 1, 2, \ldots, m \tag{1}$$

where $\mathbf{W}_i$ $(i = 1, 2, \ldots, m)$ are positive semi-definite matrices. Below it will be assumed that the non-linear perturbation class is one of the non-linear functions satisfying the inequality constraint and guaranteeing the existence and uniqueness of the corresponding non-linear systems.

Definition

The non-linear system (1) is said to be *robustly stable* if for any non-linear functions satisfying the constraints in (1) it is asymptotically stable.

For system (1) the following result was obtained.

Theorem 1

System (1) is robustly stable if there exist positive numbers $\lambda_1, \lambda_2, \ldots, \lambda_m$ such that the matrix Riccati inequality

$$\mathbf{A}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{A} + \left(\sum_{i=1}^{m} \lambda_i^{-1}\right)\mathbf{P}^2 + \sum_{i=1}^{m} \lambda_i \mathbf{W}_i < 0 \tag{2}$$

has a positive definite solution $\mathbf{P} > 0$.

Proof

Consider the following Lyapunov function candidate:

$$V[x(t), t] = x^{\mathrm{T}}(t)\mathbf{P}x(t) + \sum_{i=1}^{m} \lambda_i \int_0^t \{x^{\mathrm{T}}(s)\mathbf{W}_i x(s) - g_i^{\mathrm{T}}[x(s), s]g_i[x(s), s]\}\, \mathrm{d}s$$

Then

$$\begin{aligned}\dot{V}[x(t), t] &= \left\{\mathbf{A}x(t) + \sum_{i=1}^{m} g_i[x(t), t]\right\}^{\mathrm{T}} \mathbf{P}x(t) \\ &\quad + x^{\mathrm{T}}(t)\mathbf{P}\left\{\mathbf{A}x(t) + \sum_{i=1}^{m} g_i[x(t), t]\right\} + \sum_{i=1}^{m} \lambda_i x^{\mathrm{T}}(t)\mathbf{W}_i x(t) - \sum_{i=1}^{m} \lambda_i g_i^{\mathrm{T}}[x(t), t]g_i[x(t), t] \\ &= x^{\mathrm{T}}(t)\left\{\mathbf{A}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{A} + \sum_{i=1}^{m} \lambda_i \mathbf{W}_i + \sum_{i=1}^{m} \lambda_i^{-1} x^{\mathrm{T}}(t)\mathbf{P}^2\right\} x(t) \\ &\quad + \sum_{i=1}^{m}\left\{-\lambda_i^{-1}x^{\mathrm{T}}(t)\mathbf{P}^2 x(t) + g_i^{\mathrm{T}}[x(t), t]\mathbf{P}x(t) + x^{\mathrm{T}}(t)\mathbf{P}g_i[x(t), t] - \lambda_i g_i^{\mathrm{T}}[x(t), t]g_i[x(t), t]\right\}\end{aligned}$$

$$= \boldsymbol{x}^{\mathrm{T}}(t)\left[\mathbf{A}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{A} + \sum_{i=1}^{m} \lambda_i \mathbf{W}_i + \left(\sum_{i=1}^{m} \lambda_i^{-1}\right)\mathbf{P}^2\right]\boldsymbol{x}(t)$$

$$- \sum_{i=1}^{m} \left\{\boldsymbol{g}_i[\boldsymbol{x}(t), t]\sqrt{\lambda_i} - \sqrt{(\lambda_i^{-1})}\mathbf{P}\boldsymbol{x}(t)\right\}^{\mathrm{T}} \left\{\boldsymbol{g}_i[\boldsymbol{x}(t), t]\sqrt{\lambda_i} - \sqrt{(\lambda_i^{-1})}\mathbf{P}\boldsymbol{x}(t)\right\}$$

$$\leqslant \boldsymbol{x}^{\mathrm{T}}(t)\left[\mathbf{A}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{A} + \sum_{i=1}^{m} \lambda_i \mathbf{W}_i + \left(\sum_{i=1}^{m} \lambda_i^{-1}\right)\mathbf{P}^2\right]\boldsymbol{x}(t)$$

from which it can be seen that $\dot{V}[\boldsymbol{x}(t), t] < 0$ for any $\boldsymbol{x}(t) \neq 0$. From the Lyapunov theory it is established that system (1) is robustly stable, which completes the proof.

This theorem can be used to obtain many simple criteria for robust stability.

Corollary 1

The system

$$\dot{\boldsymbol{x}}(t) = \mathbf{A}\boldsymbol{x}(t) + \sum_{i=1}^{m} \boldsymbol{g}_i[\boldsymbol{x}(t), t]$$

$$\|\boldsymbol{g}_i[\boldsymbol{x}(t), t]\|_2 \leqslant \beta_i \|\boldsymbol{x}(t)\|_2, \qquad i = 1, 2, \ldots, m$$

is robustly stable if

$$\sum_{i=1}^{m} \beta_i < \frac{1}{2\lambda_{\mathrm{M}}(\mathbf{P})}$$

where $\mathbf{A}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{A} = -\mathbf{I}$.

Proof

For the given $\mathbf{P}$, the matrix Riccati inequality becomes

$$-\mathbf{I} + \left(\sum_{i=1}^{m} \lambda_i^{-1}\right)\mathbf{P}^2 + \sum_{i=1}^{m} \lambda_i \beta_i^2 \mathbf{I} < 0$$

which is equivalent to

$$-1 + \left(\sum_{i=1}^{m} \lambda_i^{-1}\right)\lambda_{\mathrm{M}}^2(\mathbf{P}) + \sum_{i=1}^{m} \lambda_i \beta_i^2 < 0$$

Choosing $\lambda_i = \lambda_{\mathrm{M}}(\mathbf{P})/\beta_i$, it is possible to obtain

$$\sum_{i=1}^{m} \beta_i < 1/[2\lambda_{\mathrm{M}}(\mathbf{P})]$$

This completes the proof using Theorem 1.

When $m = 1$, Corollary 1 is reduced to the following result, which is obtained in [**11**] using a different approach. Pandolfi and Zwart [**19**] showed that this is also valid for linear distributed parameter systems.

Corollary 2

Consider the system

$$\dot{\boldsymbol{x}}(t) = [\mathbf{A} + \Delta\mathbf{A}(t)]\boldsymbol{x}(t), \qquad \|\Delta\mathbf{A}(t)\|_2 \leqslant \alpha$$

This system is robustly stable if

$$\alpha < \frac{1}{2\lambda_{\mathrm{M}}(\mathbf{P})}$$

where $\mathbf{P}$ is the positive definite solution of the Lyapunov equation $\mathbf{A}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{A} = -\mathbf{I}$.

A similar argument as in the proof of Corollary 1, where $\lambda_i = \lambda_{\mathrm{M}}(\mathbf{P})/\sqrt{[\lambda_{\mathrm{M}}(\mathbf{W}_i)]}$, leads to the following result.

Corollary 3

System (1) is robustly stable if there exists a positive definite matrix $\mathbf{Q}$ such that

$$\sum_{i=1}^{m} \sqrt{[\lambda_{\mathrm{M}}(\mathbf{W}_i)]} < \frac{\lambda_m(\mathbf{Q})}{2\lambda_{\mathrm{M}}(\mathbf{P})} \tag{3}$$

where $\mathbf{P}$ is the positive definite solution of the Lyapunov equation

$$\mathbf{A}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{A} = -\mathbf{Q}$$

Moreover, the best bound in (3) can be achieved when $\mathbf{Q} = \mathbf{I}$.

Combining Lemma 2 and Theorem 1, the following corollary is obtained.

Corollary 4

The system

$$\dot{\boldsymbol{x}}(t) = \left(\mathbf{A} + \sum_{i=1}^{p} \mathbf{B}_i\right)\boldsymbol{x}(t) + \sum_{i=1}^{m} \boldsymbol{g}_i[\boldsymbol{x}(t), t]$$

$$\boldsymbol{g}_i^{\mathrm{T}}[\boldsymbol{x}(t), t]\boldsymbol{g}_i[\boldsymbol{x}(t), t] \leqslant \boldsymbol{x}^{\mathrm{T}}(t)\mathbf{W}_i\boldsymbol{x}(t), \qquad i = 1, 2, \ldots, m$$

is robustly stable if there exist positive numbers $\lambda_1, \lambda_2, \ldots, \lambda_m$ and positive definite matrices $\mathbf{R}_1, \mathbf{R}_2, \ldots, \mathbf{R}_p$ such that the matrix inequality

$$\mathbf{A}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{A} + \sum_{i=1}^{p} \mathbf{P}\mathbf{B}_i\mathbf{R}_i^{-1}\mathbf{B}_i^{\mathrm{T}}\mathbf{P} + \left(\sum_{i=1}^{m} \lambda_i^{-1}\right)\mathbf{P}^2 + \sum_{i=1}^{m} \lambda_i \mathbf{W}_i + \sum_{i=1}^{p} \mathbf{R}_i < 0$$

has a positive definite solution $\mathbf{P} > 0$.

Proof

From Theorem 1, the system is robustly stable if

$$\mathbf{A}^{\mathrm{T}}\mathbf{P}+\mathbf{P}\mathbf{A}+\sum_{i=1}^{p}(\mathbf{P}\mathbf{B}_i+\mathbf{B}_i^{\mathrm{T}}\mathbf{P})+\left(\sum_{i=1}^{m}\lambda_i^{-1}\right)\mathbf{P}^2+\sum_{i=1}^{m}\lambda_i\mathbf{W}_i<0$$

Applying Lemma 2, the proof can be completed.

Next a study is made of the robust stability of linear systems with structured perturbation:

$$\dot{x}(t)=\left(\mathbf{A}+\sum_{i=1}^{m}k_i\mathbf{B}_i\right)x(t)+\sum_{i=1}^{p}g_i[x(t),t] \qquad (4)$$
$$g_i^{\mathrm{T}}(x,t)g(x,t)\leqslant x^{\mathrm{T}}\mathbf{W}_i x$$

where $k\overset{\mathrm{def}}{=}(k_1,k_2,\ldots,k_m)$ denotes the structured uncertainty. This system was studied in reference [**6**] with no non-linear terms. Similar results are obtained from Theorem 1 for system (4).

Theorem 2

Let $\mathbf{P}$ denote the positive definite solution of the Lyapunov equation $\mathbf{A}^{\mathrm{T}}\mathbf{P}+\mathbf{P}\mathbf{A}=-\mathbf{Q}$, and let

$$\mathbf{P}_i=\mathbf{B}_i^{\mathrm{T}}\mathbf{P}+\mathbf{P}\mathbf{B}_i,\qquad \mathbf{P}_{\mathrm{e}}=[\mathbf{P}_1\,\mathbf{P}_2\cdots\mathbf{P}_m]\qquad i=1,2,\ldots,m$$

System (4) is robustly stable if there exist positive numbers $\lambda_1,\lambda_2,\ldots,\lambda_p$ such that one of the following conditions holds:

$$\sum_{i=1}^{m}k_i^2<\frac{\lambda_m^2[\mathbf{Q}-(\Sigma_{i=1}^{p}\lambda_i^{-1})\mathbf{P}^2-\Sigma_{i=1}^{p}\lambda_i\mathbf{W}_i]}{\sigma_{\mathrm{M}}^2(\mathbf{P}_{\mathrm{e}})} \qquad (5)$$

$$\sum_{i=1}^{m}|k_i|<\frac{\lambda_m[\mathbf{Q}-(\Sigma_{i=1}^{p}\lambda_i^{-1})\mathbf{P}^2-\Sigma_{i=1}^{p}\lambda_i\mathbf{W}_i]}{\max_{1\leqslant i\leqslant m}\sigma_{\mathrm{M}}(\mathbf{P}_i)} \qquad (6)$$

$$|k_i|<\frac{\lambda_m[\mathbf{Q}-(\Sigma_{i=1}^{p}\lambda_i^{-1})\mathbf{P}^2-\Sigma_{i=1}^{p}\lambda_i\mathbf{W}_i]}{\Sigma_{i=1}^{m}\sigma_{\mathrm{M}}(\mathbf{P}_i)} \qquad (7)$$

Proof of condition (5)

For system (4), the matrix inequality (2) becomes

$$-\mathbf{Q}+\sum_{i=1}^{m}k_i\mathbf{P}_i+\sum_{i=1}^{m}\lambda_i^{-1}\mathbf{P}^2+\sum_{i=1}^{p}\lambda_i\mathbf{W}_i<0 \qquad (8)$$

Since

$$\sum_{i=1}^{m}k_i\mathbf{P}_i=[k_1\mathbf{I}\,k_2\mathbf{I}\cdots k_m\mathbf{I}][\mathbf{P}_1^{\mathrm{T}}\,\mathbf{P}_2^{\mathrm{T}}\cdots\mathbf{P}_m^{\mathrm{T}}]^{\mathrm{T}}$$

it follows that

$$\lambda_{\mathrm{M}}\left(\sum_{i=1}^{m}k_i\mathbf{P}_i\right)\leqslant\left\|\sum_{i=1}^{m}k_i\mathbf{P}_i\right\|\leqslant\sigma_{\mathrm{M}}([k_1\mathbf{I}\,k_2\mathbf{I}\cdots k_m\mathbf{I}])\sigma_{\mathrm{M}}([\mathbf{P}_1^{\mathrm{T}}\,\mathbf{P}_2^{\mathrm{T}}\cdots\mathbf{P}_m^{\mathrm{T}}]^{\mathrm{T}})=\sqrt{\left(\sum_{i=1}^{m}k_i^2\right)}\sigma_{\mathrm{M}}(\mathbf{P}_{\mathrm{e}})$$

Taking this into inequality (8), condition (5) guarantees (8).

Proof of condition (6)

Since

$$\left\|\sum_{i=1}^{m}k_i\mathbf{P}_i\right\|_2\leqslant\sum_{i=1}^{m}|k_i|\,\|\mathbf{P}_i\|_2\leqslant\left(\sum_{i=1}^{m}|k_i|\right)\max_{1\leqslant i\leqslant m}\sigma_{\mathrm{M}}(\mathbf{P}_i)$$

by taking this into inequality (8) it is possible to obtain condition (6).

Proof of condition (7)

Since

$$\left\|\sum_{i=1}^{m}k_i\mathbf{P}_i\right\|_2\leqslant\sum_{i=1}^{m}|k_i|\,\|\mathbf{P}_i\|_2\leqslant\max_{1\leqslant i\leqslant m}|k_i|\left[\sum_{i=1}^{m}\sigma_i(\mathbf{P}_i)\right]$$

by taking this into inequality (8) it is possible to obtain condition (7). This completes the proof.

Choosing specific values of λ_i, many sufficient conditions for robust stability criteria can be obtained. The following is one of them, where $\lambda_i=\lambda_{\mathrm{M}}(\mathbf{P})/\sqrt{[\lambda_{\mathrm{M}}(\mathbf{W}_i)]}$.

Corollary 5

As in Theorem 2, system (4) is robustly stable if one of the following conditions holds:

$$\sum_{i=1}^{m}k_i^2<\frac{[\lambda_m(\mathbf{Q})-2\{\Sigma_{i=1}^{p}\sqrt{[\lambda_{\mathrm{M}}(\mathbf{W}_i)]}\}\lambda_{\mathrm{M}}(\mathbf{P})]^2}{\sigma_{\mathrm{M}}^2(\mathbf{P}_{\mathrm{e}})} \qquad (9)$$

$$\sum_{i=1}^{m}|k_i|<\frac{\lambda_m(\mathbf{Q})-2\{\Sigma_{i=1}^{p}\sqrt{[\lambda_{\mathrm{M}}(\mathbf{W}_i)]}\}\lambda_{\mathrm{M}}(\mathbf{P})}{\max_{1\leqslant i\leqslant m}\sigma_{\mathrm{M}}(\mathbf{P}_i)} \qquad (10)$$

$$|k_i|<\frac{\lambda_m(\mathbf{Q})-2\{\Sigma_{i=1}^{p}\sqrt{[\lambda_{\mathrm{M}}(\mathbf{W}_i)]}\}\lambda_{\mathrm{M}}(\mathbf{P})}{\Sigma_{i=1}^{m}\sigma_{\mathrm{M}}(\mathbf{P}_i)} \qquad (11)$$

Remark

With no non-linear perturbation, condition (9) with $\mathbf{Q}=\mathbf{I}$ is obtained by Zhou and Khargonekar [**13**]. The results here can be further improved from inequality (8) by the technique given in reference [**16**], but this will be left to the reader.

Finally, a study is made of the robust stability of linear systems with unstructured perturbation. Consider

$$\dot{x}(t)=\left(\mathbf{A}+\sum_{i=1}^{m}\Delta\mathbf{B}_i\right)x(t)+\sum_{i=1}^{p}g_i[x(t),t] \qquad (12a)$$

$$\Delta\mathbf{B}_i\Delta\mathbf{B}_i^{\mathrm{T}}\leqslant\mathbf{\Omega}_i,\qquad g_i^{\mathrm{T}}(x,t)g_i(x,t)\leqslant x^{\mathrm{T}}\mathbf{W}_i x \qquad (12b)$$

For this system the following result is obtained.

Theorem 3

System (12) is robustly stable if one of the following conditions holds:

1. There exist positive numbers $r_1, r_2, \ldots, r_m, \lambda_1, \lambda_2, \ldots, \lambda_p$ such that the matrix inequality

$$\mathbf{A}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{P}\left(\sum_{i=1}^{p} \lambda_i^{-1} + \sum_{i=1}^{m} r_i^{-1}\boldsymbol{\Omega}_i\right)\mathbf{P} + \sum_{i=1}^{p} \lambda_i \mathbf{W}_i + \left(\sum_{i=1}^{m} r_i\right)\mathbf{I} < 0 \qquad (13)$$

has a positive definite solution $\mathbf{P} > 0$.

2. There exists a positive definite matrix $\mathbf{Q}$ such that

$$\sum_{i=1}^{m} \surd[\lambda_{\mathrm{M}}(\mathbf{P}\boldsymbol{\Omega}_i\mathbf{P})] + \sum_{i=1}^{p} \lambda_{\mathrm{M}}(\mathbf{P})\surd[\lambda_{\mathrm{M}}(\mathbf{W}_i)] < \tfrac{1}{2}\lambda_m(\mathbf{Q})$$

where $\mathbf{P}$ is the positive definite solution of $\mathbf{A}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{A} = -\mathbf{Q}$.

3. $\mathbf{A}$ is stable and there exist positive numbers $r_1, r_2, \ldots, r_m, \lambda_1, \lambda_2, \ldots, \lambda_p$ such that

$$\|\mathbf{C}^{\mathrm{T}}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\|_\infty < 1$$

where

$$\mathbf{B}\mathbf{B}^{\mathrm{T}} = \sum_{i=1}^{m} r_i^{-1}\boldsymbol{\Omega}_i + \sum_{i=1}^{p} \lambda_i^{-1}\mathbf{I}$$

$$\mathbf{C}^{\mathrm{T}}\mathbf{C} = \sum_{i=1}^{p} \lambda_i \mathbf{W}_i + \left(\sum_{i=1}^{m} r_i\right)\mathbf{I}$$

Proof of condition 1

From Corollary 3, system (12) is asymptotically stable if there exist positive numbers $r_1, r_2, \ldots, r_m, \lambda_1, \lambda_2, \ldots, \lambda_p$ such that the matrix inequality

$$\mathbf{A}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{P}\left(\sum_{i=1}^{m} r_i^{-1}\Delta\mathbf{B}_i\Delta\mathbf{B}_i^{\mathrm{T}}\right)\mathbf{P} + \left(\sum_{i=1}^{p} \lambda_i^{-1}\right)\mathbf{P}^2 + \sum_{i=1}^{p} \lambda_i \mathbf{W}_i + \left(\sum_{i=1}^{m} r_i\right)\mathbf{I} < 0$$

Applying (12b), condition 1 is obtained.

Proof of condition 2

Let $\mathbf{P}$ be the positive definite solution of $\mathbf{A}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{A} = -\mathbf{Q}$. The matrix inequality in condition 1 can be guaranteed if

$$-\lambda_m(\mathbf{Q}) + \sum_{i=1}^{m} r_i^{-1}\lambda_{\mathrm{M}}(\mathbf{P}\boldsymbol{\Omega}_i\mathbf{P}) + \sum_{i=1}^{m} r_i + \sum_{i=1}^{p} \lambda_i^{-1}\lambda_{\mathrm{M}}^2(\mathbf{P}) + \sum_{i=1}^{m} \lambda_{\mathrm{M}}(\mathbf{W}_i) < 0$$

Choosing $r_i = \surd[\lambda_{\mathrm{M}}(\mathbf{P}\boldsymbol{\Omega}_i\mathbf{P})]$ and $\lambda_i = \lambda_{\mathrm{M}}(\mathbf{P})/\surd[\lambda_{\mathrm{M}}(\mathbf{W}_i)]$, condition 2 can be obtained.

Proof of condition 3

This is straightforward from condition 1 and Lemma 1.

4 ILLUSTRATIVE EXAMPLES

The two examples presented in this section will show how to use the test criteria developed in this paper.

Example 1

Consider the system

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= (\mathbf{A} + k\mathbf{B})\boldsymbol{x}(t) + f[\boldsymbol{x}(t), t] \\ \|f[\boldsymbol{x}(t), t]\|_2 &\leqslant \beta\|\boldsymbol{x}(t)\|_2, \qquad |k| \leqslant \alpha \end{aligned} \qquad (14)$$

where

$$\mathbf{A} = \begin{pmatrix} -3 & 1 \\ -1 & -5 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 0.5 & 0 \\ 0.75 & 1 \end{pmatrix}$$

The Lyapunov equation $\mathbf{P}\mathbf{A} + \mathbf{A}^{\mathrm{T}}\mathbf{P} = -\mathbf{I}$ has the positive definite solution

$$\mathbf{P} = \begin{pmatrix} 0.1641 & 0.0078 \\ 0.0078 & 0.1016 \end{pmatrix}$$

From Corollary 5,

$$|k| < \frac{1 - 2\beta\lambda_{\mathrm{M}}(\mathbf{P})}{\sigma_{\mathrm{M}}(\mathbf{P}\mathbf{B} + \mathbf{B}^{\mathrm{T}}\mathbf{P})} = \frac{1 - 0.33\beta}{0.2784}$$

which can be guaranteed if $0.2784\alpha + 0.33\beta < 1$. Therefore, if $0.2784\alpha + 0.33\beta < 1$, system (14) is robustly stable.

Figure 1 shows the trajectory of the system for $\beta = 0$ with k varying from -3.5 to 3.6. From the above discussion, the stability range obtained is $|k| < 1/0.2784 = 3.592$. This is also shown in Fig. 1: when $|k| \leqslant 3.5$, the system is indeed robustly stable. However, for $k = 3.6$, the trajectory diverges and hence the system is not asymptotically stable. The upper bound obtained for robust stability is indeed a very good estimate.

Example 2

Consider the system

$$\dot{\boldsymbol{x}}(t) = (\mathbf{A} + \Delta\mathbf{B})\boldsymbol{x}(t) + f[\boldsymbol{x}(t), t]$$

$$\Delta\mathbf{B}\Delta\mathbf{B}^{\mathrm{T}} \leqslant \begin{pmatrix} \alpha_1^2 & 0 \\ 0 & \alpha_2^2 \end{pmatrix}$$

$$f(\boldsymbol{x}, t)^{\mathrm{T}} f(\boldsymbol{x}, t) \leqslant \begin{pmatrix} \beta_1^2 & 0 \\ 0 & \beta_2^2 \end{pmatrix}$$

where

$$\mathbf{A} = \begin{pmatrix} -2 & 0 \\ 1 & -3 \end{pmatrix}$$

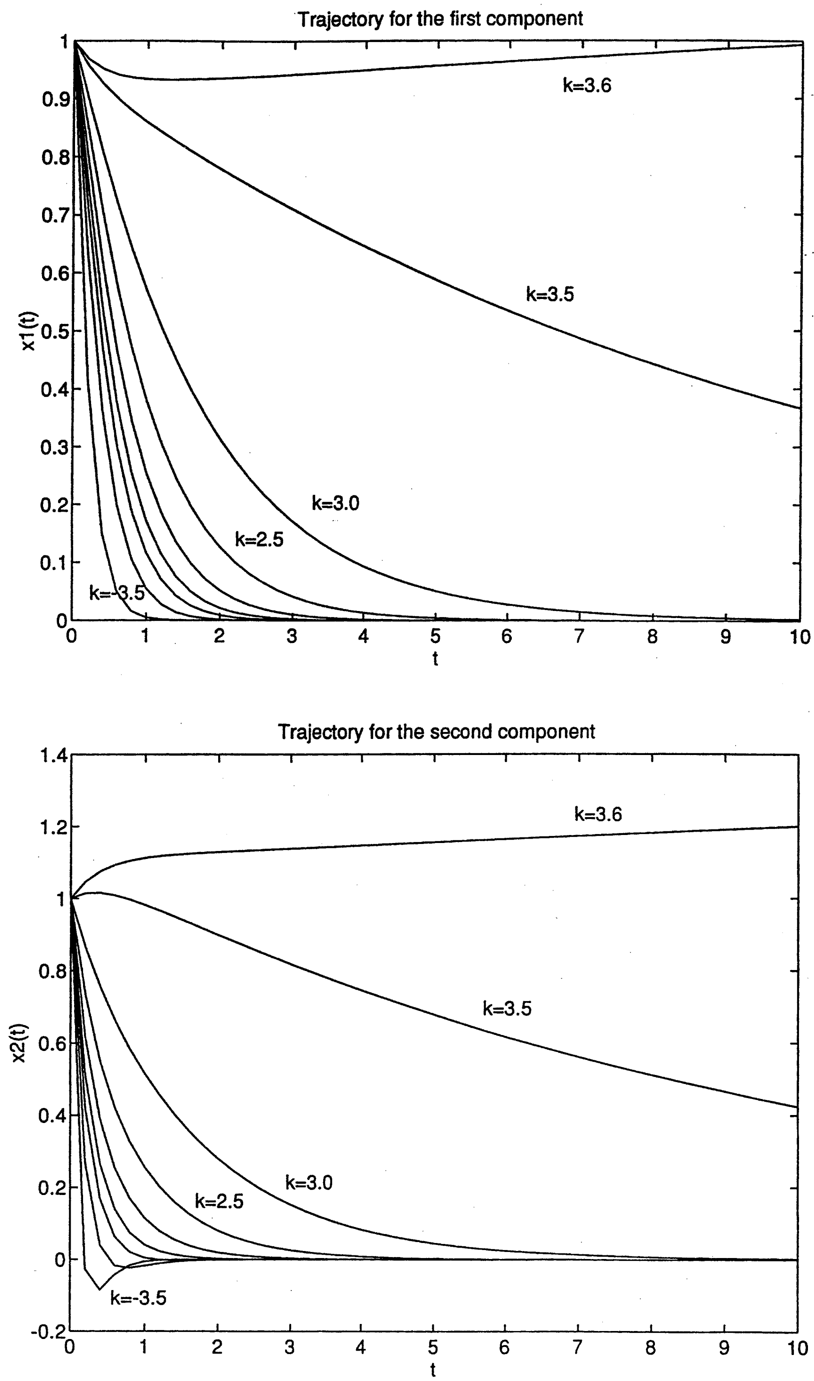

Fig. 1 State trajectories for various values of uncertainty parameter k

From $\mathbf{A}^{\mathrm{T}}\mathbf{P} + \mathbf{PA} = -\mathbf{I}$,

$$\mathbf{P} = \begin{pmatrix} 0.2667 & 0.0333 \\ 0.0333 & 0.1667 \end{pmatrix}$$

and hence from Theorem 3, condition 2,

$$\sqrt{[\lambda_{\mathrm{M}}(\mathbf{\Omega P}^2)]} + \max(\beta_1, \beta_2)\lambda_{\mathrm{M}}(\mathbf{P}) < \tfrac{1}{2} \qquad (15)$$

where

$$\mathbf{\Omega} = \begin{pmatrix} \alpha_1^2 & 0 \\ 0 & \alpha_2^2 \end{pmatrix}$$

Since $\lambda_{\mathrm{M}}(\mathbf{\Omega P}^2) \leqslant \mathrm{tr}(\mathbf{\Omega P}^2)$, by taking this into inequality (15) it can be concluded that if

$$\sqrt{(0.0722\alpha_1^2 + 0.0289\alpha_2^2)} + 0.2768\max(\beta_1, \beta_2) < 0.5$$

the system is robustly stable. If $\lambda_M(\mathbf{\Omega P}^2) \leqslant \lambda_M(\mathbf{\Omega})\lambda_M(\mathbf{P}^2)$ is used, it is found that if

$$\max(\alpha_1, \alpha_2) + \max(\beta_1, \beta_2) < \frac{0.5}{\lambda_M(\mathbf{P})} = \frac{0.5}{0.2768} = 1.8064$$

the system is robustly stable.

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