Robust stability of uncertain linear systems with multiple time delays

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Abstract: In this paper, the robust stability of uncertain linear systems with multiple time delays is studied. Sufficient conditions for robust stability of linear time delay systems with convex perturbations are obtained. From these sufficient conditions, a few results on robust stability of systems with other perturbations are derived. Some previously known sufficient conditions are generalized.

Keywords: stability, time delay systems, matrix measure, robustness, perturbations

NOTATION

$\mathbf{A}^+$	elementwise upper bound matrix for matrix $\mathbf{A}$
$\mathbf{A}^-$	elementwise lower bound matrix for matrix $\mathbf{A}$
$\bar{\mathbf{A}}$	$(\mathbf{A}^+ + \mathbf{A}^-)/2$
$\exp(s)$	exponential function e^s
$\mathbf{I}$	identity matrix
j	$\sqrt{-1}$
$\|\mathbf{X}\|$	vector or matrix norm of X
$\|X\|_e$	elementwise absolute value of X
$X \leqslant_e Y$	elementwise inequalities between X and Y
$\lambda(\mathbf{A})$ or $\lambda_i(\mathbf{A})$	any eigenvalue of matrix $\mathbf{A}$
$\mu(\mathbf{A})$	matrix measure of matrix $\mathbf{A}$
$\mu_p(\mathbf{A})$	matrix measure induced by the p norm where $p = 1, 2, +\infty$

1 INTRODUCTION

In recent years there has been great interest in the stability analysis of time delay systems. Time delay is often encountered in various engineering systems, such as chemical processes and long transmission lines in pneumatic, hydraulic and rolling mill systems. It is well known that the time delays may destabilize the system; therefore the stability of such systems becomes a very important issue for practical applications.

Robust stability of linear systems has attracted considerable attention [see references (**1**) to (**6**)]. Recently, a great deal of research work has been done on robust stability of time delay systems. The result of Brierley *et al.* (**7**) has inspired many researchers in the last ten years. They obtained a relationship between a linear time delay system and a Lyapunov equation: the linear time delay system is asymptotically stable independent of delay if and only if the Lyapunov equation has a positive definite solution. Thus, matrix measure has been used to study the stability and robust stability of linear time delay systems and many interesting results have been obtained (**8**–**12**). Robust stability of linear time delay systems with linear or nonlinear perturbations is also investigated (**13**–**17**) using the Lyapunov second method.

This paper studies the robust stability of linear systems with multiple delays and with parameter perturbations. Using a general matrix measure, some sufficient conditions are obtained for robust stability against parameter perturbations and time delays. The paper is organized as follows. In Section 2, preliminaries are given. The main results are presented in the third section. A few illustrative examples are given in Section 4. The paper is concluded in the final section.

2 PRELIMINARIES

Let $\|x\|$ denote a vector norm of x on C^n, and $\|\mathbf{A}\|$ the induced matrix norm of $\mathbf{A}$ given the vector norm $\|\cdot\|$. $\mu(\mathbf{A})$ is the matrix measure of $\mathbf{A}$ defined as

$$\mu(\mathbf{A}) \triangleq \lim_{\theta \downarrow 0^+} \frac{\|\mathbf{I} + \theta\mathbf{A}\| - 1}{\theta}$$

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where **I** is the identity matrix. The matrix **A** is stable if its eigenvalues have negative real parts. The matrix measure gives an upper bound for the magnitude of the solution of a differential equation $\dot{x}(t) = \mathbf{A}(t)x(t)$, i.e. the following well-known Coppel inequality (**18**):

$$\|x(t)\| \leqslant \|x(t_0)\| \exp\left(\int_{t_0}^{t} \mu(\mathbf{A}(\tau))\, \mathrm{d}\tau\right)$$

which renders it suitable for stability investigations.

Lemma 1 (**18, 19**)

$\mu(\mathbf{A})$ is well defined for any norm and has the following properties:

(a) μ is convex on $C^{n\times n}$, i.e. for any $\alpha_j \geqslant 0$ $(1 \leqslant j \leqslant k)$ and $\sum_{j=1}^{k} \alpha_j = 1$, and any matrices $\mathbf{A}_j$ $(1 \leqslant j \leqslant k)$,

$$\mu\left(\sum_{j=1}^{k} \alpha_j \mathbf{A}_j\right) \leqslant \sum_{j=1}^{k} \alpha_j \mu(\mathbf{A}_j)$$

(b) For any norm and any **A**,

$$-\|\mathbf{A}\| \leqslant -\mu(-\mathbf{A}) \leqslant \operatorname{Re} \lambda(\mathbf{A}) \leqslant \mu(\mathbf{A}) \leqslant \|\mathbf{A}\|$$

(c) For the 1-norm, $\|\boldsymbol{x}\|_1 = \sum_{i=1}^{n} \|\boldsymbol{x}_i\|$, the induced matrix measure μ_1 is given by

$$\mu_1(\mathbf{A}) = \max_j \left[\operatorname{Re}(a_{jj}) + \sum_{i\neq j} |a_{ij}|\right]$$

For the 2-norm, $\|\boldsymbol{x}\|_2 = (\sum_{i=1}^{n} |\boldsymbol{x}_i|^2)^{1/2}$, the induced matrix measure μ_2 is given by

$$\mu_2(\mathbf{A}) = \max_i \left[\lambda_i(\mathbf{A} + \mathbf{A}^*)/2\right]$$

For the ∞-norm, $\|\boldsymbol{x}\|_\infty = \max_{1\leqslant i\leqslant n} |\boldsymbol{x}_i|$, the induced matrix measure μ_∞ is given by

$$\mu_\infty(\mathbf{A}) = \max_i \left[\operatorname{Re}(a_{ii}) + \sum_{j\neq i} |a_{ij}|\right]$$

(d) For any non-singular matrix **T** and any vector norm $\|\cdot\|$ with the induced matrix measure μ, then $\|\mathbf{T}\boldsymbol{x}\|$ defines another vector norm and its induced matrix measure μ_{T} is given by

$$\mu_{\mathrm{T}}(\mathbf{A}) = \mu(\mathbf{T}\mathbf{A}\mathbf{T}^{-1})$$

(e) Let $\mathscr{N}$ be the set of all vector norms on C^n. For any $\rho \in \mathscr{N}$, the corresponding matrix measure is denoted as μ_ρ. Let $\mathscr{P}$ be the set of non-singular matrices in $C^{n\times n}$ and $\mathscr{H}$ be the set of positive Hermitian matrices in $C^{n\times n}$. Using μ_P^p and μ_H to denote the matrix measures induced by $\|\boldsymbol{x}\|_P^p = \|\mathbf{P}\boldsymbol{x}\|_p$ $(p = 1, 2, \infty)$ and $\|\boldsymbol{x}\|_H = \sqrt{(x^* Hx)}$, respectively, then

$$\max_{1\leqslant i\leqslant n} \operatorname{Re} \lambda_i(\mathbf{A}) = \inf_{\rho\in\mathscr{N}} \mu_\rho(\mathbf{A}) = \inf_{P\in\mathscr{P}} \mu_P^1(\mathbf{A}) = \inf_{P\in\mathscr{P}} \mu_P^2(\mathbf{A}) = \inf_{P\in\mathscr{P}} \mu_P^\infty(\mathbf{A}) = \inf_{H\in\mathscr{H}} \mu_H(\mathbf{A})$$

(f) **A** is stable if and only if there exists a matrix measure μ such that $\mu(\mathbf{A}) < 0$.

Lemma 2

Consider the delayed linear system

$$\dot{x}(t) = \mathbf{A}x(t) + \sum_{i=1}^{m} \mathbf{B}_i x(t - \tau_i) \tag{1}$$

for

$$x(t) = \phi(t), \qquad t \leqslant 0$$

where **A** and $\mathbf{B}_i$ are square matrices, the time delays τ_i $(i = 1, 2, \ldots, m)$ are non-negative constants and $\phi(t)$ is the initial condition. The system (1) is asymptotically stable independent of time delay if and only if

$$\dot{w}(t) = \left(\mathbf{A} + \sum_{i=1}^{m} z_i \mathbf{B}_i\right) w(t) \tag{2}$$

is asymptotically stable for any $\omega_i \in [0, 2\pi]$ where $z_i = \exp(\mathrm{j}\omega_i)$ $(i = 1, 2, \ldots, m)$ and $\mathrm{j} = \sqrt{-1}$, i.e. the following matrices are stable:

$$\mathbf{A} + \sum_{i=1}^{m} z_i \mathbf{B}_i, \qquad z_i = \exp(\mathrm{j}\omega_i) \tag{3}$$

for

$$\omega_i \in [0, 2\pi], \qquad i = 1, 2, \ldots, m$$

Proof. This can be proved from the result of Brierley *et al.* (**7**) and the Lyapunov theorem [Theorem 2.2.1 in reference (**20**)]. See also Tissir and Hmamed (**12**) for the single-delay case.

3 MAIN RESULTS

Consider the following uncertain linear systems with multiple time delays:

$$\dot{x}(t) = \mathbf{A}_0 x(t) + \mathbf{B}_0 x(t-\tau_0) + \sum_{i=1}^{m} \alpha_i \mathbf{A}_i x(t) + \sum_{i=1}^{r} \beta_i \mathbf{B}_i x(t-\tau_i) \tag{4}$$

for

$$x(t) = \phi(t), \qquad t \leqslant 0$$

$$\alpha_i \geqslant 0, \qquad \sum_{i=1}^{m} \alpha_i = 1$$

$$\beta_i \geqslant 0, \qquad \sum_{i=1}^{r} \beta_i = 1$$

where $\mathbf{A}_0$, $\mathbf{B}_0$, $\mathbf{A}_1, \ldots, \mathbf{A}_m$, $\mathbf{B}_1, \ldots, \mathbf{B}_r$ are square real matrices and the time delays τ_i $(i = 1, 2, \ldots, r)$ are non-negative numbers. The nominal system for (4) is

$$\dot{x}(t) = \mathbf{A}_0 x(t) + \mathbf{B}_0 x(t-\tau_0) \tag{5}$$

for

$$x(t) = \phi(t), \qquad t \leqslant 0$$

The system (4) can be regarded as the nominal system (5) with convex parameter perturbation. It will be shown that such a kind of perturbation characterizes a very general class of perturbation and implies many previously known models in the current literature. The main objective in this paper is to give sufficient conditions for robust stability of the perturbed time delay system (4).

In the current literature, only the matrix measures induced by 1-norm, ∞-norm and 2-norm are used to obtain some sufficient conditions for stability or robust stability of linear time delay systems [references (**8**) to (**12**) and references therein]. However, as noted by Fang *et al.* (**19**), the general matrix measure defined in Section 2 can be used to study the stability of linear and non-linear systems, and more general sufficient conditions for stability can be obtained. Therefore, it is expected that this can be generalized to the stability of systems (4) and (5), giving the following theorem.

Theorem 1

The system (4) is robustly stable independent of time delays if there exists a matrix measure $\mu(\cdot)$ such that

$$\mu(\mathbf{A}_0 + \exp(\mathrm{j}\omega_0)\mathbf{B}_0) + \max_{1\leqslant i\leqslant m} \mu(\mathbf{A}_i) + \max_{1\leqslant i\leqslant r} \mu(\exp(\mathrm{j}\omega_i)\mathbf{B}_i) < 0 \tag{6}$$

for any $\omega_i \in [0, 2\pi]$ $(i = 0, 1, 2, \ldots, r)$.

Proof. From Lemma 2, the system (4) is asymptotically stable independent of time delays if and only if for any $\omega_i \in [0, 2\pi]$ $(i = 0, 1, 2, \ldots, r)$, the matrix

$$\mathbf{X} \stackrel{\text{def}}{=} \mathbf{A}_0 + \exp(\mathrm{j}\omega_0)\mathbf{B}_0 + \sum_{i=1}^{m} \alpha_i \mathbf{A}_i + \sum_{i=1}^{r} \beta_i \exp(\mathrm{j}\omega_i)\mathbf{B}_i \tag{7}$$

is stable. From Lemma 1(f), $\mathbf{X}$ is stable if and only if there exists a matrix measure $\mu(\cdot)$ such that $\mu(\mathbf{X}) < 0$. From the convexity of the matrix measure [Lemma 1(a)],

$$\begin{aligned}\mu(\mathbf{X}) &\leqslant \mu(\mathbf{A}_0 + \exp(\mathrm{j}\omega_0)\mathbf{B}_0) + \sum_{i=1}^{m} \alpha_i \mu(\mathbf{A}_i) + \sum_{i=1}^{r} \beta_i \mu(\exp(\mathrm{j}\omega_i)\mathbf{B}_i) \\ &\leqslant \mu(\mathbf{A}_0 + \exp(\mathrm{j}\omega_0)\mathbf{B}_0) + \left(\sum_{i=1}^{m} \alpha_i\right) \max_{1\leqslant i\leqslant m} \mu(\mathbf{A}_i) + \left(\sum_{i=1}^{r} \beta_i\right) \max_{1\leqslant i\leqslant r} \mu(\exp(\mathrm{j}\omega_i)\mathbf{B}_i) \\ &= \mu(\mathbf{A}_0 + \exp(\mathrm{j}\omega_0)\mathbf{B}_0) + \max_{1\leqslant i\leqslant m} \mu(\mathbf{A}_i) + \max_{1\leqslant i\leqslant r} \mu(\exp(\mathrm{j}\omega_i)\mathbf{B}_i)\end{aligned}$$

Thus, if (6) holds, then $\mu(\mathbf{X}) < 0$. This completes the proof.

From this theorem, many sufficient conditions for stability or robust stability of linear time delay systems can be obtained. In what follows, this point will be illustrated.

Corollary 1

The system (4) is robustly stable independent of time delays if there exists a matrix measure $\mu(\cdot)$ such that

$$\mu(\mathbf{A}_0) + \max_{1\leqslant i\leqslant m} \mu(\mathbf{A}_i) + \|\mathbf{B}_0\| + \max_{1\leqslant i\leqslant r} \|\mathbf{B}_i\| < 0$$

Proof. From Lemma 1(b),

$$\mu(\exp(\mathrm{j}\omega_i)\mathbf{B}_i) \leqslant \|\exp(\mathrm{j}\omega_i)\mathbf{B}_i\| \leqslant |\exp(\mathrm{j}\omega_i)|\|\mathbf{B}_i\| = \|\mathbf{B}_i\|$$

from which the proof can easily be completed.

If $\mathbf{A}_i = 0$ and $\mathbf{B}_i = 0$, then the system (4) is reduced to the nominal system (5). The following corollary is obtained.

Corollary 2

The system (5) is asymptotically stable independent of time delay if there exists a matrix measure $\mu(\cdot)$ such that

$$\mu(\mathbf{A}_0 + \exp(j\omega)\mathbf{B}_0) < 0, \qquad \text{for any } \omega \in [0, 2\pi]$$

or

$$\mu(\exp(j\omega)\mathbf{B}_0) < -\mu(\mathbf{A}_0), \qquad \text{for any } \omega \in [0, 2\pi]$$

By specifying the matrix measure $\mu(\cdot)$, the previously known stability conditions are obtained.

Corollary 3 (**8–12**)

The system (5) is asymptotically stable if either

$$\mu_p(\mathbf{A}_0 + \exp(j\omega)\mathbf{B}_0) < 0, \qquad \omega \in [0, 2\pi]$$

or

$$\mu_p(\mathbf{A}_0) + \mu_p(\exp(j\omega)\mathbf{B}_0) < 0, \qquad \omega \in [0, 2\pi]$$

for $p = 1$ or $p = 2$ or $p = \infty$.

When $m = r = 1$, the system (4) becomes the linear time delay system with unstructured perturbation, which was studied by Tissir and Hmamed (**12**). From Theorem 1, the following generalization is obtained.

Corollary 4

The system

$$\dot{x}(t) = (\mathbf{A} + \mathbf{E})x(t) + (\mathbf{B} + \mathbf{F})x(t - \tau) \tag{8}$$

where $\mathbf{E}$ and $\mathbf{F}$ represent the perturbations matrices is asymptotically stable independent of time delay if there exists a matrix measure $\mu(\cdot)$ if either

$$\mu(\mathbf{A} + \exp(j\omega)\mathbf{B}) + \|\mathbf{E}\| + \|\mathbf{F}\| < 0, \qquad \omega \in [0, 2\pi]$$

or

$$\mu(\mathbf{A}) + \mu(\exp(j\omega)\mathbf{B}) + \|\mathbf{E}\| + \|\mathbf{F}\| < 0, \qquad \omega \in [0, 2\pi]$$

Let $|\cdot|_e$ denote the elementwise absolute value and $\leqslant_e$ the elementwise inequalities. If the perturbations $\mathbf{E}$ and $\mathbf{F}$ have upper bounds, then the following result is obtained.

Corollary 5

The system (8) with the following perturbations

$$|\mathbf{E}|_e \leqslant_e \mathbf{U}_e, \qquad |\mathbf{F}|_e \leqslant_e \mathbf{U}_f$$

is robustly stable independent of delay if either of the next conditions is satisfied:

$$\mu_p(\mathbf{A} + \exp(j\omega)\mathbf{B}) + \mu_p(\mathbf{U}_e) + \mu_p(\mathbf{U}_f) < 0, \qquad \omega \in [0, 2\pi]$$

$$\mu_p(\mathbf{A}) + \mu_p(\exp(j\omega)\mathbf{B}) + \mu_p(\mathbf{U}_e) + \mu_p(\mathbf{U}_f) < 0, \qquad \omega \in [0, 2\pi]$$

where $p = 1$ or $p = 2$ or $p = \infty$.

Proof. It can be seen that $|\exp(j\omega)\mathbf{F}|_e = |\mathbf{F}|_e$. For $p = 1$ and $p = \infty$, from Lemma 1(c),

$$\mu_p(\mathbf{E}) \leqslant \mu_p(\mathbf{U}_e), \qquad \mu_p(\exp(j\omega)\mathbf{F}) \leqslant \mu_p(\mathbf{U}_f)$$

From Theorem 1, this corollary can be easily proved for $p = 1, \infty$. Let $\rho(\cdot)$ denote the spectral radius. From Horn and Johnson (**21**), for any matrices $\mathbf{C}$ and $\mathbf{D}$, if $|\mathbf{C}|_e \leqslant_e \mathbf{D}$, then $\rho(\mathbf{C}) \leqslant \rho(|\mathbf{C}|_e) \leqslant \rho(\mathbf{D})$. Thus,

$$\mu_2(\mathbf{E}) \leqslant \mu_2(\mathbf{U}_e), \qquad \mu_2(\exp(j\omega)\mathbf{F}) \leqslant \mu_2(\mathbf{U}_f)$$

From Theorem 1, the proof is completed.

Since $\mu_p(\mathbf{U}_e) \leqslant \|\mathbf{U}_e\|_p$ and $\mu_p(\mathbf{U}_f) \leqslant \|\mathbf{U}_f\|_p$, Corollary 5 is better than Theorem 4 in Tissir and Hmamed (**12**). The other results by Tissir and Hmamed (**12**) can be improved in a similar way.

As noted in Fang *et al.* (**19, 22**), the matrix measures $\mu_p(\cdot)$ $(p = 1, 2, \infty)$ themselves may be conservative for a certain structure of system matrices. The matrix measures introduced in Lemma 1(d) and (e) can be used to reduce such conservatism. For any non-singular matrix $\mathbf{T}$, from Lemma 1(d) and (e) and for any matrix $\mathbf{C}$,

$$\mu_{\mathbf{T}}^p(\mathbf{C}) = \mu_p(\mathbf{T}\mathbf{C}\mathbf{T}^{-1}), \qquad p = 1, 2, \infty \tag{9}$$

From this and Theorem 1, the following very useful result is formalized.

Theorem 2

The system (4) is robustly stable independent of time delays if there exists a non-singular matrix $\mathbf{T}$ such that

$$\mu_p(\mathbf{T}\mathbf{A}_0\mathbf{T}^{-1} + \exp(j\omega_0)\mathbf{T}\mathbf{B}_0\mathbf{T}^{-1}) + \max_{1\leqslant i\leqslant m} \mu_p(\mathbf{T}\mathbf{A}_i\mathbf{T}^{-1}) + \max_{1\leqslant i\leqslant r} \mu_p(\exp(j\omega_i)\mathbf{T}\mathbf{B}_i\mathbf{T}^{-1}) < 0 \tag{10}$$

for any $\omega_i \in [0, 2\pi]$ $(i = 0, 1, 2, \ldots, r)$, where $p = 1$ or 2 or ∞.

The robust stability of system (4) with the interval perturbations $\alpha_i \in [-h_i, h_i]$ and $\beta_i \in [-v_i, v_i]$, where $\mathbf{A}_0$ is an $\mathbf{M}$ matrix and $\mathbf{A}_i$ and $\mathbf{B}_i$ are non-negative (or non-positive) matrices, are studied by Luo *et al.* (**16**). In the next theorem, a sufficient condition for the stability of such model is given.

Theorem 3

Consider the linear time delay system

$$\dot{x}(t) = \mathbf{A}_0 x(t) + \mathbf{B}_0 x(t-\tau_0) + \sum_{i=1}^{m} \alpha_i \mathbf{A}_i x(t) + \sum_{i=1}^{r} \beta_i \mathbf{B}_i x(t-\tau_i) \tag{11}$$

for

$$x(t) = \phi(t), \qquad t \leqslant 0,$$

$$\alpha_i \in [\alpha_i^-, \alpha_i^+], \qquad \beta_i \in [-\beta_i^-, \beta_i^+]$$

The system (11) is robustly stable independent of time delays if there exists a matrix measure $\mu(\cdot)$ such that

$$\sum_{i=1}^{m} \max\{\mu(\alpha_i^- \mathbf{A}_i), \mu(\alpha_i^+ \mathbf{A}_i)\} + \sum_{i=1}^{r} \max\{\mu(\exp(\mathrm{j}\omega_i)\beta_i^- \mathbf{B}_i), \mu(\exp(\mathrm{j}\omega_i)\beta_i^+ \mathbf{B}_i)\} < -\mu(\mathbf{A}_0 + \exp(\mathrm{j}\omega_0)\mathbf{B}_0)$$

for any $\omega_i \in [0, 2\pi]$ $(i = 0, 1, 2, \ldots, r)$.

Proof. Since $\alpha_i \in [\alpha_i^-, \alpha_i^+]$ and $\beta_i \in [\beta_k^-, \beta_k^+]$, there exists an $l_i \in [0, 1]$ and $p_k \in [0, 1]$ such that

$$\alpha_i = l_i \alpha_i^- + (1 - l_i)\alpha_i^+, \qquad \beta_k = p_k \beta_k^- + (1 - p_k)\beta_k^+$$

Following a similar procedure as in the proof of Theorem 1, the proof can easily be completed.

Using the matrix measures in system (10), from Theorem 3 the following result can be obtained.

Corollary 6

The system (11) is robustly stable independent of time delays if there exists a non-singular matrix $\mathbf{T}$ such that

$$\mu(\mathbf{T}\mathbf{A}_0\mathbf{T}^{-1} + \exp(\mathrm{j}\omega_0)\mathbf{T}\mathbf{B}_0\mathbf{T}^{-1}) + \sum_{i=1}^{m} \max\{\mu_p(\alpha_i^- \mathbf{T}\mathbf{A}_i\mathbf{T}^{-1}), \mu_p(\alpha_i^+ \mathbf{T}\mathbf{A}_i\mathbf{T}^{-1})\} + \sum_{i=1}^{r} \max\{\mu_p(\exp(\mathrm{j}\omega_i)\beta_i^- \mathbf{T}\mathbf{B}_i\mathbf{T}^{-1}), \mu_p(\exp(\mathrm{j}\omega_i)\beta_i^+ \mathbf{T}\mathbf{B}_i\mathbf{T}^{-1})\} < 0$$

for any $\omega_i \in [0, 2\pi]$ $(i = 0, 1, 2, \ldots, r)$, where $p = 1$ or $p = 2$ or $p = \infty$.

Tissir and Hmamed (**12**) also studied the linear time delay systems with interval parameters and obtained some interesting sufficient conditions for robust stability. The following result deals with interval linear systems with multiple delays.

Theorem 4

Consider the following linear system with multiple delays:

$$\dot{x}(t) = \mathbf{A}x(t) + \sum_{i=1}^{m} \mathbf{B}_i x(t - \tau_i) \tag{12}$$

for

$$x(t) = \phi(t), \qquad t \leqslant 0$$

where $\tau_i \geqslant 0$ $(i = 1, 2, \ldots, m)$ and

$$\mathbf{A}^- = (a_{ij}^-) \leqslant_{\mathrm{e}} \mathbf{A} = (a_{ij}) \leqslant_{\mathrm{e}} \mathbf{A}^+ = (a_{ij}^+)$$

$$\mathbf{B}_k^- = (b_{ij}^{(k)-}) \leqslant_{\mathrm{e}} \mathbf{B}_k = (b_{ij}^{(k)}) \leqslant_{\mathrm{e}} \mathbf{B}_k^+ = (b_{ij}^{(k)+}), \qquad k = 1, 2, \ldots, m \tag{13}$$

Let the vertex matrices be denoted by

$$\mathscr{A} = \{\mathbf{X} = (x_{ij}) | x_{ij} = a_{ij}^- \text{ or } a_{ij}^+\}$$

$$\mathscr{B}_{\mathrm{k}} = \{\mathbf{Y} = (y_{ij}) | y_{ij} = b_{ij}^{(k)-} \text{ or } b_{ij}^{(k)+}\}, \qquad k = 1, 2, \ldots, m \tag{14}$$

The system (12) and (13) is robustly stable independent of time delays if there exists a matrix measure $\mu(\cdot)$ such that

$$\max_{\mathbf{X}\in\mathscr{A}} \mu(\mathbf{X}) + \sum_{k=1}^{m} \max_{\mathbf{Y}\in\mathscr{B}_k} \mu(\exp(\mathrm{j}\omega_k)\mathbf{Y}) < 0 \tag{15}$$

for any $\omega_k \in [0, 2\pi]$ $(k = 1, 2, \ldots, m)$.

Proof. Let $p = 2^{n^2}$, the number of matrices in either $\mathscr{A}$ or $\mathscr{B}_i$. For any $\mathbf{A}$ and $\mathbf{B}_i$, there exist numbers $\lambda_i \geqslant 0$ and $\gamma_i^{(k)} \geqslant 0$ $(i = 1, 2, \ldots, p$ and $k = 1, 2, \ldots, m)$ satisfying $\sum_{i=1}^{m} \lambda_i = 1$ and $\sum_{i=1}^{p} \gamma_i^{(k)} = 1$ $(k = 1, 2, \ldots, m)$ such that

$$\mathbf{A} = \sum_{i=1}^{p} \lambda_i \mathbf{M}_i, \qquad \mathbf{B}_k = \sum_{i=1}^{m} \gamma_i^{(k)} \mathbf{N}_i^{(k)}$$

where

$$\mathscr{A} = \{\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_p\}$$

$$\mathscr{B}_k = \{\mathbf{N}_1^{(k)}, \mathbf{N}_2^{(k)}, \ldots, \mathbf{N}_p^{(k)}\}$$

Following the same procedure as in the proof of Theorem 1, the proof can easily be completed.

Following the procedure of Tissir and Hmamed (**12**), the next result can also be obtained.

Theorem 5

The system (12) and (13) is robustly stable independent of delays if

$$\mu_p\left((\mathbf{A}^+ + \mathbf{A}^-)/2 + \sum_{i=1}^{m} \exp(\mathrm{j}\omega_i)(\mathbf{B}_i^+ + \mathbf{B}_i^-)/2\right) + \mu_p((\mathbf{A}^+ - \mathbf{A}^-)/2) + \sum_{i=1}^{m} \mu_p((\mathbf{B}_i^+ - \mathbf{B}_i^-)/2) < 0 \quad (16)$$

for any $\omega_i \in [0, 2\pi]$ $(i = 1, 2, \ldots, m)$, where $p = 1$ or 2 or ∞.

Proof. Let $\bar{\mathbf{A}} = (\mathbf{A}^+ + \mathbf{A}^-)/2$ and $\bar{\mathbf{B}}_i = (\mathbf{B}_i^+ + \mathbf{B}_i^-)/2$. Then for any $\mathbf{A}$ and $\mathbf{B}_i$ satisfying (13),

$$|\mathbf{A} - \bar{\mathbf{A}}|_e \leqslant_e (\mathbf{A}^+ - \mathbf{A}^-)/2$$

$$|\mathbf{B}_i - \bar{\mathbf{B}}_i|_e \leqslant_e (\mathbf{B}_i^+ - \mathbf{B}_i^-)/2$$

From this and the proof of Corollary 5,

$$\mu_p(\mathbf{A} - \bar{\mathbf{A}}) \leqslant \mu_p((\mathbf{A}^+ - \mathbf{A}^-)/2)$$

$$\mu_p(\mathbf{B}_i - \bar{\mathbf{B}}_i) \leqslant \mu_p((\mathbf{B}_i^+ - \mathbf{B}_i^-)/2)$$

Thus,

$$\begin{aligned}
&\mu_p\left(\mathbf{A} + \sum_{i=1}^{m} \exp(\mathrm{j}\omega_i)\mathbf{B}_i\right) \\
&\quad + \mu_p\left(\bar{\mathbf{A}} + \sum_{i=1}^{m} \exp(\mathrm{j}\omega_i)\bar{\mathbf{B}}_i + (\mathbf{A} - \bar{\mathbf{A}}) + \sum_{i=1}^{m} \exp(\mathrm{j}\omega_i)(\mathbf{B}_i - \bar{\mathbf{B}}_i)\right) \\
&\leqslant \mu_p\left(\bar{\mathbf{A}} + \sum_{i=1}^{m} \exp(\mathrm{j}\omega_i)\bar{\mathbf{B}}_i\right) + \mu_p(\mathbf{A} - \bar{\mathbf{A}}) + \sum_{i=1}^{m} \mu_p(\exp(\mathrm{j}\omega_i)(\mathbf{B}_i - \bar{\mathbf{B}}_i)) \\
&\leqslant \mu_p\left(\bar{\mathbf{A}} + \sum_{i=1}^{m} \exp(\mathrm{j}\omega_i)\bar{\mathbf{B}}_i\right) + \mu_p(|\mathbf{A} - \bar{\mathbf{A}}|_e) + \sum_{i=1}^{m} \mu_p(|\exp(\mathrm{j}\omega_i)(\mathbf{B}_i - \bar{\mathbf{B}}_i)|_e) \\
&\leqslant \mu_p\left(\bar{\mathbf{A}} + \sum_{i=1}^{m} \exp(\mathrm{j}\omega_i)\bar{\mathbf{B}}_i\right) + \mu_p((\mathbf{A}^+ - \mathbf{A}^-)/2) + \sum_{i=1}^{m} \mu_p((\mathbf{B}_i^+ - \mathbf{B}_i^-)/2)
\end{aligned}$$

from which and the assumption, the proof is complete.

When $m = 1$, this theorem gives an improved version of Theorem 6 in Tissir and Hmamed (**12**). In this paper, only the robust stability of linear time delay systems is studied. The sufficient conditions for robust stability with decay rate can easily be obtained in a similar way by noticing the fact that $x(t)$ converges to zero with decay rate α if and only if $y(t) = \exp(\alpha t)x(t)$ converges to zero. Some results are already given in references (**9**) to (**12**). The reader is encouraged to formalize the corresponding results for robust stability with decay rate from the results above.

As a final thought, how to reduce the conservativeness of the results is discussed. The matrix measure chosen really affects the applicability of the results because the matrix measure depends on the structure of the matrices involved. As shown in Theorem 2, the conservative nature can be reduced if an appropriate state transform is applied before the main results of Theorem 1. A minimization problem for the left-hand side of system (10) can be formalized for the choice of **T**. Many sufficient conditions can be obtained from this theorem; for example, when **T** is chosen to be a positive diagonal matrix, a Gershgorin-type sufficient condition can be obtained which also leads to sufficient conditions in terms of the **M** matrix. This issue will be discussed elsewhere.

Of course, these results are still conservative in the sense that the stability is guaranteed independent of time delays. It may be helpful to establish similar results for the systems with time-varying delays or to find some conditions that can be expressed in terms of certain knowledge of delays such as upper bounds. This will be investigated in the future.

4 ILLUSTRATIVE EXAMPLES

In this section, a few examples are presented to illustrate the usefulness of the results.

Example 1 (**12**)

Consider the system (8) where

$$\mathbf{A} = \begin{pmatrix} -4 & 0 \\ 0 & -3 \end{pmatrix}, \qquad \mathbf{B} = a\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

$$\mathbf{E} = \begin{pmatrix} 0.3 & -0.2 \\ 0.5 & 0.1 \end{pmatrix}, \qquad \mathbf{F} = \begin{pmatrix} -0.1 & 0.25 \\ 0 & 0.2 \end{pmatrix}$$

where a is a scalar parameter. Let

$$\mathbf{B}_0 = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

The matrix measure induced by the 2-norm in Corollary 5 is used. The second condition in Corollary 5 can be guaranteed if

$$\mu_2(\mathbf{A}) + |a|\mu_2(\mathbf{B}_0) + \mu_2(|\mathbf{E}|_2) + \mu_2(|\mathbf{F}|_e) < 0$$

from which it is found that $|a| < 0.9747$. From the result of Tissir and Hmamed (**12**),

$$|a| < \frac{-\mu_2(\mathbf{A}) - \||\mathbf{E}|_e\|_2 - \||\mathbf{F}|_e\|_2}{\mu_2(\mathbf{B}_0)} = 0.8986$$

This shows that Corollary 5 is an improvement over the result in Tssir and Hmamed (**12**).

Example 2 (**12**)

Consider the interval time delay system (**12**) and (**13**) with $m = 1$ and

$$\mathbf{A}^- = \begin{pmatrix} -7.2 & 1 \\ 2 & -7 \end{pmatrix}, \qquad \mathbf{A}^+ = \begin{pmatrix} -6.5 & 7 \\ 2.5 & -6.3 \end{pmatrix}$$

$$\mathbf{B}^- = \begin{pmatrix} 0.35 & -1 \\ -1.6 & -0.4 \end{pmatrix}, \qquad \mathbf{B}^+ = \begin{pmatrix} 1 & 0 \\ -0.5 & 0.6 \end{pmatrix}$$

The average matrices and the average difference matrices are

$$\frac{\mathbf{A}^+ + \mathbf{A}^-}{2} = \begin{pmatrix} -6.85 & 4 \\ 2.25 & -6.65 \end{pmatrix}$$

$$\frac{\mathbf{B}^+ + \mathbf{B}^-}{2} = \begin{pmatrix} 0.675 & -0.5 \\ -1.05 & 0.1 \end{pmatrix}$$

$$\frac{\mathbf{A}^+ - \mathbf{A}^-}{2} = \begin{pmatrix} 0.35 & 3 \\ 0.25 & 0.35 \end{pmatrix}$$

$$\frac{\mathbf{B}^+ - \mathbf{B}^-}{2} = \begin{pmatrix} 0.325 & 0.5 \\ 0.55 & 0.5 \end{pmatrix}$$

Numerical computation gives

$$\max_{\omega\in[0,2\pi]} \mu_2((\mathbf{A}^+ + \mathbf{A}^-)/2 + \exp(\mathrm{j}\omega)(\mathbf{B}^+ + \mathbf{B}^-)/2)) = -3.3055$$

and $\mu_2((\mathbf{A}^+ - \mathbf{A}^-)/2) + \mu_2((\mathbf{B}^+ - \mathbf{B}^-)/2) = 2.9197$ and $\|(\mathbf{A}^+ - \mathbf{A}^-)/2\|_2 + \|(\mathbf{B}^+ - \mathbf{B}^-)/2\|_2 = 3.9893$. The left-hand side of (16) is equal to -0.3858 and, from Theorem 5, the system (12) and (13) is robustly stable. However, from Tissir and Hmamed (**12**),

$$\max_{\omega\in[0,2\pi]} \mu_2((\mathbf{A}^+ + \mathbf{A}^-)/2 + \exp(\mathrm{j}\omega)(\mathbf{B}^+ + \mathbf{B}^-)/2) + \|(\mathbf{A}^+ - \mathbf{A}^-)/2\|_2 + \|(\mathbf{B}^+ - \mathbf{B}^-)/2\|_2 = 0.6838$$

so the results in Tissir and Hmamed (**12**) cannot be used.

Example 3

Consider the system

$$\dot{x}(t) = \mathbf{A}x(t) + \mathbf{B}x(t - \tau) \tag{17}$$

where

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} -1.8 & -1.4 \\ 2.3 & 1.9 \end{pmatrix}$$

Since

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} 1.8 & 2.4 \\ -4.3 & -4.9 \end{pmatrix}$$

and from Lemma 1(c),

$$\mu_1(\mathbf{A} - \mathbf{B}) = 6.1, \qquad \mu_\infty(\mathbf{A} - \mathbf{B}) = 4.2$$

$$\mu_2(\mathbf{A} - \mathbf{B}) = 1.9321$$

it can be concluded that $\mu_p(\mathbf{A} + \exp(\mathrm{j}\omega)\mathbf{B}) < 0$ is not true for $\omega = \pi$ and $p = 1, 2, \infty$. Hence, all previously known results in terms of matrix measure cannot be applied. If

$$\mathbf{T} = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$$

is chosen then

$$\mathbf{TAT}^{-1} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}, \qquad \mathbf{TBT}^{-1} = \begin{pmatrix} -0.4 & 0.5 \\ 0 & 0.5 \end{pmatrix}$$

For any $\omega \in [0, 2\pi]$,

$$\mu_1(\mathbf{TAT}^{-1} + \exp(\mathrm{j}\omega)\mathbf{TBT}^{-1}) = \max\{-1 - 0.4\cos\omega, -2 + \cos\omega\} < 0$$

From Theorem 2, it can be concluded that the system (17) is asymptotically stable independent of time delay. This example shows that previously known results are conservative, and by doing a state transformation before those results are applied, the conservativeness of the results can be reduced.

5 CONCLUSIONS

The robust stability of uncertain linear systems with multiple delays has been studied. Using the matrix measure technique, a few sufficient conditions have been established for robust stability independent of time delays. It may be possible to generalize this technique to obtain some less conservative sufficient conditions for robust stability which depend on the time delays or upper bounds of time delays. This issue is still under investigation.

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