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A THEOREM ON THE k-ADIC REPRESENTATION OF POSITIVE INTEGERS

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Abstract. In this paper, a theorem on the asymptotic property of a summation of digits in a k-adic representation is presented.

Let $k > 1$ be a fixed integer. Then any positive integer x can be uniquely represented by the following k-adic form:

$$x = a_1 k^{n_1} + a_2 k^{n_2} + \cdots + a_t k^{n_t}, \tag{1}$$

where $n_1 > n_2 > \cdots > n_t \geq 0$ are integers and $a_1, a_2, \ldots, a_t$ are nonnegative integers not exceeding $k-1$. Define

$$\alpha(x) = \sum_{i=1}^{t} a_i, \qquad A(x) = \sum_{y \leq x} \alpha(y). \tag{2}$$

In 1940, Bush ([1]) showed that

$$A(x) = \frac{k-1}{2\log k} x \log x + o(x \log x), \tag{3}$$

where log denotes the natural logarithm. In 1948, Bellman and Shapiro ([2]) improved this result and proved that

$$A(x) = \frac{k-1}{2\log k} x \log x + O(x \log\log x) \tag{4}$$

for $k = 2$. In 1949, Mirsky ([3]) showed that the O-term can be replaced by $O(x)$ for any $k \geq 2$. In 1955, Cheo and Yien ([4]) gave another proof for the result and obtained:

$$A(x) = \frac{k-1}{2\log k} x \log x + O(x), \tag{5}$$

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and proved that $O(x)$ cannot be replaced by $O(x^t)$ for any fixed $t < 1$. Their proof relies on the identity

$$A(x) = \frac{n_1(k-1)}{2}\sum_{i=1}^{t} a_i k^{n_i} - \frac{k-1}{2}\sum_{i=1}^{t}(n_1 - n_i)a_i k^{n_i} + \frac{1}{2}\sum_{i=1}^{t} a_i(a_i - 1)k^{n_i} + \sum_{i=1}^{t} a_i + \sum_{i=1}^{t}\left(\sum_{j=1}^{i-1} a_j\right) a_i k^{n_i},$$

where a_i, n_i, and t are as in (1). The first sum equals $\frac{1}{2}(k-1)[\log x/\log k]x$ and the four other sums are shown to be $O(x)$ after complicated mathematical manipulations.

In this paper, we apply a different identity and obtain an estimate on the constant contained in $O(x)$, consequently providing a much simpler proof to the previously known results. The following result is obtained.

Theorem. *For any integer $k \geq 2$, we have*

$$A(x) = \frac{k-1}{2\log k} x \log x + \theta(x)x, \tag{6}$$

where

$$-\frac{5k-4}{8} \leq \theta(x) \leq \frac{k+1}{2}.$$

To prove this Theorem, we need the following result due to J. L. Lagrange:

Lemma ([5])**.**

$$\frac{n - \alpha(n)}{k-1} = \sum_{r=1}^{\infty}\left[\frac{n}{k^r}\right], \tag{7}$$

where $[a]$ denotes the integral part of the real number a.

Proof of the Theorem. Using the Lemma, we have

$$\begin{aligned} A(x) &= \sum_{n\leq x}\left(n - (k-1)\sum_{r=1}^{\infty}\left[\frac{n}{k^r}\right]\right) \\ &= \frac{1}{2}x(x+1) - (k-1)\sum_{r=1}^{\infty}\sum_{n\leq x}\left[\frac{n}{k^r}\right] \\ &= \frac{1}{2}x(x+1) - (k-1)\sum_{1\leq r\leq \log_k x}\left(\frac{1}{2}\left[\frac{x}{k^r}\right]\left(\left[\frac{x}{k^r}\right] - 1\right)k^r \right. \\ &\qquad\qquad \left. + \left[\frac{x}{k^r}\right]\left(x - \left[\frac{x}{k^r}\right]k^r + 1\right)\right) \\ &= \frac{1}{2}x(x+1) + \frac{1}{2}(k-1)\sum_{1\leq r\leq \log_k x} k^r\left[\frac{x}{k^r}\right] - (k-1)\sum_{1\leq r\leq \log_k x}\left[\frac{x}{k^r}\right] \\ &\quad - (k-1)\sum_{1\leq r\leq \log_k x}\left(x\left[\frac{x}{k^r}\right] - \frac{1}{2}\left[\frac{x}{k^r}\right]^2 k^r\right). \end{aligned} \tag{8}$$

However, we observe that

$$\sum_{1\le r\le \log_k x} k^r\left[\frac{x}{k^r}\right] = x[\log_k x] + \sum_{1\le r\le \log_k x} k^r\left(\left[\frac{x}{k^r}\right]-\frac{x}{k^r}\right)$$
$$= x\log_k x - \theta_1(x)x + \sum_{1\le r\le \log_k x} k^r\left(\left[\frac{x}{k^r}\right]-\frac{x}{k^r}\right),$$

$$\sum_{1\le r\le \log_k x}\left(x\left[\frac{x}{k^r}\right]-\frac{1}{2}\left[\frac{x}{k^r}\right]^2 k^r\right) = \frac{1}{2}\sum_{1\le r\le \log_k x}\left(\frac{x^2}{k^r}-k^r\left(\left[\frac{x}{k^r}\right]-\frac{x}{k^r}\right)^2\right)$$
$$= \frac{1}{2}x^2\sum_{1\le r\le \log_k x}\frac{1}{k^r}-\frac{1}{2}\sum_{1\le r\le \log_k x}k^r\left(\left[\frac{x}{k^r}\right]-\frac{x}{k^r}\right)^2,$$

where $0\le\theta_1(x)<1$. Taking these into (8), we obtain

$$\begin{aligned}A(x) = {}& \frac{1}{2}x(x+1)+\frac{k-1}{2}x\log_k x-\frac{k-1}{2}\theta_1(x)x-(k-1)\sum_{1\le r\le \log_k x}\left[\frac{x}{k^r}\right]\\ &-\frac{1}{2}\sum_{1\le r\le \log_k x}\left(\left\{\frac{x}{k^r}\right\}-\left\{\frac{x}{k^r}\right\}^2\right)k^r-\frac{k-1}{2}x^2\sum_{1\le r\le \log_k x}\frac{1}{k^r},\end{aligned}\tag{9}$$

where $\{a\}$ denotes the fractional part of the real number a. Using the following inequalities $0\le x-x^2\le 1/4$ $(0\le x\le 1)$ and $[a]\le a$, we can find $0\le\theta_2(x)\le 1$ and $0\le\theta_3(x)\le 1$ such that

$$\sum_{1\le r\le \log_k x}\left[\frac{x}{k^r}\right]=\theta_2(x)\frac{x}{k-1},$$
$$\sum_{1\le r\le \log_k x}\left(\left\{\frac{x}{k^r}\right\}-\left\{\frac{x}{k^r}\right\}^2\right)k^r=\theta_3(x)\frac{kx}{4(k-1)},$$
$$x^2\sum_{1\le r\le \log_k x}\frac{1}{k^r}=\frac{x^2}{k-1}-\frac{1}{k-1}\frac{x^2}{k^{[\log_k x]}}.$$

Substituting these into (9), we finally arrive at

$$\begin{aligned}A(x) &= \frac{k-1}{2}\frac{x\log x}{\log k}+\left(-\frac{k-1}{2}\theta_1(x)-\theta_2(x)+\frac{1}{2}-\frac{k}{8}\theta_3(x)+\frac{x}{2k^{[\log_k x]}}\right)x\\ &= \frac{k-1}{2}\frac{x\log x}{\log k}+\theta(x)x,\end{aligned}\tag{10}$$

where

$$-\frac{5k-4}{8}\le\theta(x)\le\frac{k+1}{2}.$$

This completes the proof.

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