# Multiple Multidimensional Knapsack Problem and Its Applications in Cognitive Radio Networks

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*Abstract*— **In this paper, a new variant of the standard knapsack problem is investigated and applied in cognitive radio networks. More specifically, the centralized spectrum allocation in cognitive radio networks is formulated as a multiple multidimensional knapsack problem. We propose an exact solution and a heuristic algorithm with guaranteed performance. The performance of the proposed algorithms are compared numerically.**

*Index Terms*— **Cognitive Radio Networks, Spectrum Allocation, Combinatorial Optimization, Online Algorithms.**

## I. Introduction

One of the most valuable resources in wireless communication is the limited frequency spectrum. Designing efficient spectrum utilization schemes is the main objective of many researchers. During the past decades, a dramatic increase of emerging wireless services has significantly aggravated the scarcity of the frequency spectrum. Meanwhile, as reported in a set of empirical studies [1], the majority of the current allocated spectrum is largely under-utilized. Therefore, an urge to utilize the ubiquitous spectrum white space is advocated. The XG program [2], for example, aims to design more dynamic and efficient spectrum access policies and regulation rules. The new schemes are termed as dynamic spectrum access (DSA) schemes. Cognitive radios, usually based on the software defined radio (SDR) techniques, are proposed as the solution to this spectrum utilization problem. A cognitive radio based user is capable of detecting the instantaneous channel condition as well as adjusting the transmission frequencies to exploit the spectrum in an efficient and economical fashion. For detailed overview of the dynamic spectrum access schemes for cognitive radio networks, refer to [3].

In the current literature, the dynamic spectrum access schemes are usually categorized into three classes: exclusive use model, open sharing model and hierarchical access model [4]. The exclusive use model represents the scenarios where every user, either primary or secondary, does not share the spectrum with others. Meanwhile, in the open sharing model, all users obey a certain spectrum access rule and share the spectrum[1]. Lastly, in the hierarchical access model, primary users and secondary users coexist and the secondary users can transmit only when the accumulated interference measured at the primary user's receiver is below a certain threshold, namely, interference temperature [2], [3]. In this paper, we follow the hierarchical access model. We consider a cognitive radio network with many primary users where each primary user possesses a primary band with a fixed bandwidth as well as a predefined interference temperature. The transmission of a secondary user on a particular primary band is permitted only if the accumulated interference does not exceed the interference temperature of the corresponding primary user.

In cognitive radio networks, one of the key techniques is the responsive detection of the spectrum availability. To ensure the accuracy of the detection, the concept of *distributed detection with centralized decision* is introduced in the literature [5]–[8]. In this framework, there is a central coordinator node (CCN), or namely, fusion center, in the range of the cognitive radio network while the joint detection is executed by all cognitive nodes collectively. The CCN collects all the reports from nodes and makes a decision according to a predefined rule. In this paper, we utilize the central node to allocate

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[1]The IEEE 802.11 protocol is a well-known example.

the available spectrum for the cognitive users in a centralized fashion. Moreover, the available spectrum needs to be allocated optimally in a sense that the aggregated throughput is maximized without violating the interference constraints. We model the spectrum allocation problem in a knapsack framework where an exact solution and a heuristic algorithm are derived.

The rest of this paper is organized as follows. Section II briefly overviews the background of the knapsack problems. The multidimensional knapsack problem for spectrum allocation is formulated in Section III where the exact and heuristic algorithms are proposed. The numerical results are provided in Section IV and the extensions of our framework are discussed in Section V. Finally, Section VI concludes this paper.

## II. Background

Knapsack problem is one of the famous challenges in combinatorial optimization. The name of knapsack is obtained by the following well-known example. Suppose a hitch-hiker needs to fill a knapsack for a trip. There are $n$ items available to select and each of them has a value of $p_i$, i.e., the degree of the usefulness of this item during the trip, as well as a size of $w_i$. A natural constraint arises that the aggregated size of all selected items cannot exceed the capacity of the knapsack, denoted by $c$. The objective of the hitch-hiker is to select a subset of items while maximizing the overall value under the capacity constraint. To model this decision process, we introduce a binary decision variable $x_i$ for each item where $x_i = 1$ if the $i$-th item is selected and packed while $x_i = 0$ otherwise. The standard knapsack problem (SKP) is given by

$\underline{SKP:}$

$$
\begin{aligned}
&\max \sum_{i=1}^{n} p_i x_i \\
&\text{s.t.} \\
&\quad \sum_{i=1}^{n} w_i x_i \leq c \\
&\quad x_i \in \{0,1\} \quad \forall i
\end{aligned} \tag{1}
$$

Although seems simple, the standard knapsack problem represents a set of non-trivial integer combinatorial optimization problems which are $\mathcal{NP}$-hard [9]. The standard knapsack problem has been studied for centuries due to its general applicability in many areas such as business [10], networking [11], cargo airline dispatching [12] and cryptography [13]. Based on the standard knapsack problem, several variants are considered in the literature. For example, a special case of the standard knapsack problem where $p_i = w_i$ is termed as the subset-sum problem and the objective is to find a subset of items whose sum is closest to but not exceeding the capacity. Another variant of the standard knapsack problem is the multiple knapsack problem. Instead of a single knapsack, we consider multiple knapsacks where each one, say $j$, has a capacity of $c_j, j = 1, \cdots, m$ and $m$ is the number of knapsacks. The decision here is not only whether to select a single item but also to which knapsack it is packed. Similarly, we introduce a binary variable of $x_{i,j} = 1$ to represent that item $i$ is selected and packed into knapsack $j$ and $x_{i,j} = 0$ otherwise. Mathematically, the multiple knapsack problem (MKP) is given by

$\underline{MKP:}$

$$
\begin{aligned}
&\max \sum_{j=1}^{m} \sum_{i=1}^{n} p_i x_{i,j} \\
&\text{s.t.} \\
&\quad \sum_{i=1}^{n} w_i x_{i,j} \leq c_j \ \forall j \\
&\quad \sum_{j=1}^{m} x_{i,j} \leq 1 \ \forall i \\
&\quad x_{i,j} \in \{0,1\} \quad \forall i,j
\end{aligned} \tag{2}
$$

where inequalities of (2) ensure that each item appears at most once in all knapsacks.

It is well known that even the simplest form of knapsack problems, i.e., the subset-sum problem, is $\mathcal{NP}$-hard. Therefore, the exact solutions are mainly focusing on dynamic programming (DP) and branch-and-bound (BB) techniques. We will follow the branch-and-bound approach in this paper.

## III. Multidimensional Knapsack Problem in Cognitive Radio Networks

### A. Formulation

We consider a cognitive radio network illustrated in Figure 1. The cognitive radio network consists of $n$ cognitive users and a centralized coordinator node (CCN) which collects the instantaneous reports from each cognitive user and decides the availability of the spectrum accordingly. We assume that there is a common channel for the communications between cognitive users

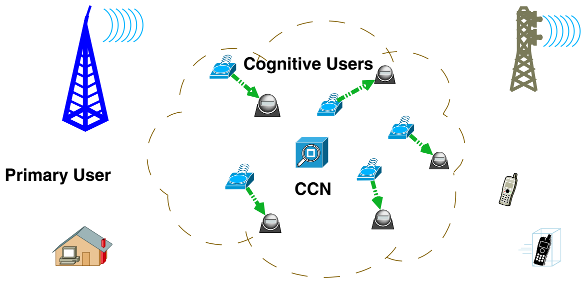


Fig. 1. Cognitive radio network.

and the CCN. Each cognitive user is a pair of transmitter and receiver with infinite backlog packets. The CCN collects the transmission requests from all cognitive users and allocate the available spectrum optimally and hence the overall throughput is maximized.

We first consider a scenario where only one primary user exists in the area. The bandwidth of the primary band is denoted by $B$. At a decision time, each cognitive user, say $i$, requests a bandwidth of $d_i$ to the CCN for the current transmission. Among all $n$ requests, the CCN decides which requests are accepted. Meanwhile, each accepted transmission generates a level of interference to the primary user. We denote this user-dependent interference as $I_i$, which is assumed to be known to the CCN by the time of decision. We will consider a more general case where $I_i$ is stochastic and unknown in Section V. The throughput is represented as a function of the bandwidth and denote this function explicitly as $h(d_i)$, which is assumed to be a nonnegative and nondecreasing function with respect to $d_i$. We denote $x_i = 1$ if the request from user $i$ is accepted and 0 otherwise. The decision problem for the CCN is given as follows.

$\underline{\mathcal{P}_1:}$

$$\max \sum_{i=1}^{n} h(d_i)x_i$$

s.t.

$$\sum_{i=1}^{n} d_i x_i \leq B \quad (3)$$

$$\sum_{i=1}^{n} I_i x_i \leq K \quad (4)$$

$$x_i \in \{0,1\} \quad \forall i \quad (5)$$

where constraint (3) ensures that the overall utilized spectrum cannot exceed the primary band, i.e., the overall available spectrum chunk. Inequality (4) indicates that the accumulated interference at the primary user is restricted by the interference temperature $K$. The CCN needs to find the optimum value, denoted by $\mathbf{x}^* = [x_1^*, \cdots, x_n^*]$ which maximizes the overall throughput while the constraints are satisfied. Therefore, the centralized spectrum allocation problem is formulated as a *two-dimensional knapsack problem*. Intuitively, the spectrum allocation problem enjoys a geometric interpretation of packing the requests in a two dimensional space where the bandwidth is one dimension and the interference temperature is the other.

We next generalize the formulation to a cognitive radio network where multiple primary users coexist. We assume that there are $m$ primary users in the area and each of them, say $j$, has a primary band with a bandwidth of $B_j$ as well as an interference temperature $K_j$. Similarly, we introduce a variable of $x_{i,j}$ which equals to 1 if the bandwidth request from the $i$-th cognitive user is accepted by assigning a frequency segment in the $j$-th primary band. Otherwise, $x_{i,j}$ equals to 0. We can formulate the decision problem of the CCN as

$\underline{\mathcal{P}_2:}$

$$\max \sum_{j=1}^{m} \sum_{i=1}^{n} h(d_i)x_{i,j}$$

s.t.

$$\sum_{i=1}^{n} d_i x_{i,j} \leq B_j \ \forall j$$

$$\sum_{i=1}^{n} I_{i,j} x_{i,j} \leq K_j \ \forall j$$

$$\sum_{j=1}^{m} x_{i,j} \leq 1 \ \forall i$$

$$x_{i,j} \in \{0,1\} \quad \forall i, j$$

It is worth noting that this is a *multiple multidimensional knapsack problem* (MMKP) which, to the best of our knowledge, has not been particularly addressed in the literature [14], [15]. Different from the general integer programming problems, MMKP always admits a feasible solution, i.e., $x_{i,j} = 0 \quad \forall i, j$. Apparently, MMKP is $\mathcal{NP}$-hard and computationally difficult to solve even in a centralized fashion. In the following, we first derive an exact solution to the MMKP problem based on decomposition and branch-and-bound techniques. Next, a heuristic algorithm with a performance guarantee is proposed.

## B. Exact Solution

We follow the branch-and-bound (BB) approach to solve the multiple multidimensional knapsack problem due to the enormous memory requirement of the alternative dynamic programming approach [15]. The branch-and-bound algorithm is in essence an intelligent enumeration of the solution space. The key challenge in BB algorithm and its variants is the determination of the upper bound. Therefore, in this section, we only focus on the derivation of the upper bound of the multiple multidimensional problem. Details of the standard branch-and-bound algorithm can be found in [16].

To attain an upper bound, one of the most common approach in BB algorithm is the linear relaxation approach where the integer constraints are dismissed. In addition, thanks to the greedy choice property [15], for a standard knapsack problem, i.e., one knapsack with only one dimension, the optimum value of the relaxed linear programming problem is easy to compute. In other words, the upper bound for a standard knapsack problem can be trivially calculated by simple greedy algorithms [9], [15]. Unfortunately, the proposed multiple multidimensional knapsack problem is remarkably complex and does not possess this property. Therefore, our objective is to decompose the complex MMKP into several simple standard knapsack problems and thus the upper bound of MMKP can be attained by solving the subproblems in a parallel fashion. Let us first consider the following problem.

$\underline{\mathcal{P}_3 :}$

$$\max_{\mathbf{x}} L(\mathbf{x}, \lambda, \gamma)$$
$$\text{s.t.} \quad \sum_{i=1}^{n} d_i x_{i,j} \le B_j \ \forall j$$
$$x_{i,j} \in \{0,1\} \ \forall i, j$$

where

$$L(\mathbf{x}, \lambda, \gamma) = \sum_{j=1}^{m}\sum_{i=1}^{n} h(d_i) x_{i,j} - \sum_{i=1}^{n} \lambda_i (\sum_{j=1}^{m} x_{i,j} - 1) - \sum_{j=1}^{m} \gamma_j (\sum_{i=1}^{n} I_{i,j} x_{i,j} - K_j) \tag{6}$$

where $\lambda_i \ge 0 \ \forall i$ and $\gamma_j \ge 0 \ \forall j$ are the dual variables. Denote the optimum value of $\mathcal{P}_3$ as $\tilde{o}$. Apparently, $\tilde{o}$ is an upper bound of the optimum value of $\mathcal{P}_2$ for arbitrary nonnegative $\lambda_i$ and $\gamma_j$. However, to achieve a tight upper bound, we would like to choose the optimum dual variables such that (6) is minimized. The dual problem is given as

$\underline{\mathcal{P}_4 :}$

$$\min_{\lambda, \gamma \ge \mathbf{0}} L(\mathbf{x}, \lambda, \gamma) \tag{7}$$

where the optimum solution can be obtained by the standard subgradient method [17]. We can rewrite (6) as

$$L(\mathbf{x}, \lambda, \gamma) = \sum_{j=1}^{m}\sum_{i=1}^{n} h(d_i) x_{i,j} - \sum_{i=1}^{n} \lambda_i (\sum_{j=1}^{m} x_{i,j} - 1) - \sum_{j=1}^{m} \gamma_j (\sum_{i=1}^{n} I_{i,j} x_{i,j} - K_j) \tag{8}$$

$$= \sum_{j=1}^{m}\sum_{i=1}^{n} h(d_i) x_{i,j} - \sum_{j=1}^{m}\sum_{i=1}^{n} \lambda_i x_{i,j} - \sum_{j=1}^{m}\sum_{i=1}^{n} \gamma_j I_{i,j} x_{i,j} + \sum_{i=1}^{n} \lambda_i + \sum_{j=1}^{m} \gamma_j K_j \tag{9}$$

$$= \sum_{j=1}^{m} \left\{ \sum_{i=1}^{n} (h(d_i) - \lambda_i - \gamma_j I_{i,j}) x_{i,j} \right\} + \sum_{i=1}^{n} \lambda_i + \sum_{j=1}^{m} \gamma_j K_j. \tag{10}$$

It is straightforward to verify that the upper bound of the original multiple multidimensional knapsack problem can be attained by calculating the following $m$ decomposed standard knapsack problems in parallel, which significantly curtails the computational burden. Define $\tilde{h}_j = h(d_i) - \lambda_i - \gamma_j I_{i,j}$. The $j$-th subproblem, i.e., the $j$-th decomposed standard knapsack problem, is given as

$$\max \sum_{i=1}^{n} \tilde{h}_j x_{i,j}$$
$$\text{s.t.} \quad \sum_{i=1}^{n} I_{i,j} x_{i,j} \le B_j$$
$$x_{i,j} \in \{0,1\} \ \forall i, j \tag{11}$$

We denote the maximum value of the $j$-th subproblem as $o_j$. The upper bound of the original multiple multidimensional knapsack problem is obtained by

$$\sum_{j=1}^{m} o_j + \sum_{i=1}^{n} \lambda_i + \sum_{j=1}^{m} \gamma_j K_j \tag{12}$$

which can be solved efficiently due to the greedy choice property possessed by the standard knapsack problem. Note that the issue of achieving the optimal dual variables, i.e., solving $\mathcal{P}_4$, as well as the rest of the branch-and-bound algorithm follow the standard procedure and thus are omitted.

### *C. Heuristic Algorithm*

It is well-known that even for the one-dimensional multiple knapsack problems, no fully polynomial-time approximation scheme (FPTAS) [18], [19] exists. In this section, we propose a heuristic algorithm to solve the multiple multidimensional knapsack problem. Following the seminal paper of [20], we define the *efficiency* of the bandwidth request from the $i$-th cognitive user as

$$e_i = \frac{h(d_i)}{\sum_{j=1}^{m} \{I_{i,j}[\sum_{i=1}^{n} I_{i,j} - K_j]^+ + d_i[\sum_{i=1}^{n} d_i - B_j]^+\}} \tag{13}$$

where $[x]^+$ denotes $\max(x, 0)$. Note that $e_i$ is positive and finite since we assume that either $\sum_{i=1}^{n} I_{i,j} > K_j$ or $\sum_{i=1}^{n} d_i > B_j$ for some primary band $j$. Otherwise, the solution is trivial by accepting all bandwidth requests.

Next, for each primary band, say $j$, at a particular time instance $t$, we define the *residue* of this band, denoted by $r_j(t)$, as

$$r_j(t) = \alpha_j(t) \times \beta_j(t) \tag{14}$$

where $\alpha_j(t)$ and $\beta_j(t)$ denote the remaining capacity and the current interference tolerance of the $j$-th primary user at time $t$, respectively. The heuristic algorithm for the multiple multidimensional knapsack problem is described as follows.

**Heuristic Algorithm:**

**Initialization:**

- Assign each request from the cognitive users with an efficiency value calculated by (13).
- Maintain a list and record all the efficiency values.

**Do:**

- Select a request in the list with the highest efficiency value.
- For this particular request, if there are a set of primary bands that can accommodate it while the capacity and the interference constraints are satisfied, choose the primary band with the minimum residue value defined in (14). Decline the request otherwise.
- Remove the selected request from the list.

**Until:**

- No request can be accepted without violating the constraints.

*Theorem 1:* The proposed heuristic algorithm has a relative performance ratio of $\frac{1}{2}$ and it is tight.

The proof of Theorem 1 generalizes the analysis of one-dimensional multiple knapsack problems in [15] and thus is omitted. We will demonstrate the efficacy of the proposed heuristic algorithm numerically in the following section.

## IV. Numerical Results

We consider a cognitive radio network sited in a $k$-by-$k$ square area. There are four primary users and their receivers are located at the four vertices of the square. The bandwidths of the primary bands are given as $[15, 20, 30, 40]$ MHz. The interference temperatures are assumed to be 1 mW for all primary users. There are five cognitive users in the area. The transmitters of the cognitive users are randomly located in the area and the receivers are randomly located 2 distance units away. The traffic demands are assumed to be $[3, 6, 13, 9, 15]$ MHz and the transmission powers are fixed as 100 mW. The noise power is 2 mW and we assume that the received power is inversely proportional to the square of the distance between two nodes. Without loss of generality, we consider

$$h(d_i) = d_i \log(1 + SNR) \tag{15}$$

and $k = 10$. We execute the exact algorithm and the heuristic algorithm for 1000 sample runs. The sample paths of the performance curves are shown partially in Figure 2. The topmost curve is the upper bound of the overall throughput, i.e., the overall throughput if all requests are accepted. The curve with $\circ$ denotes the overall throughput attained by the exact solution based on the decomposition and the branch-and-bound techniques. The curve with $\times$ represents the performance achieved by the heuristic algorithm. We also plot the half of the performance of the exact solution in a dashed line for comparison purpose. It can be observed from Figure 2 that in most scenarios, the proposed heuristic algorithm yields identical performance with the global optimum solution obtained by the exact solution with much less computational complexity. The average overall throughput calculated by the exact solution is $102.53$ Mbps whereas that of the heuristic algorithm is $94.6$ Mbps which is approximately $92\%$ of the global optimum solution.

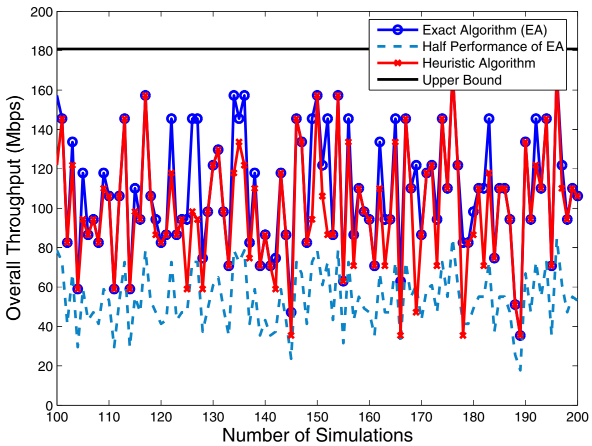


Fig. 2. Performance of the proposed algorithms.

## V. Extensions

### A. Stochastic Issue

Thus far, we have assumed that the values of $h(d_i)$ and $I_{i,j}$ are known to the CCN for the spectrum allocation. However, in practice, the values may be unknown and stochastic. Techniques in stochastic optimization can be utilized to solve this stochastic version of MMKP. For example, if the throughput, i.e., $h(d_i)$ is random, the CCN can optimize the expected overall throughput of the network by replacing the objective function. It is worth noting that the randomness of $I_{i,j}$ is more common and interesting in practice. Even the transmission power of each cognitive user is fixed, the interference perceived by the primary user is time-varying and stochastic in nature due to the channel variations. To harness the stochastic interference constraints, two lines of efforts have been made. The first category of solutions introduces chance constraints and follows the standard stochastic optimization techniques [21]–[23]. More specifically, the interference constraint is replaced by

$$Pr(\sum_{i=1}^{n} I_{i,j}x_{i,j} \leq K_j) \geq 1-\epsilon, \quad \forall j \qquad (16)$$

where $\epsilon$ is a tunable and arbitrarily small positive number. The other line of research follows the robust optimization techniques [24], where the random parameter, e.g., the interference level in our case, is assumed to be within a certain error range and the worst case performance is maximized. Therefore, our proposed MMKP framework can be extended to stochastic MMKP by replacing the interference constraints with the chance constraints or following the robust approach such as [25].

### B. Online Issue

In this paper, we consider a scenario where all the cognitive users are constantly backlogged. However, a more realistic scenario is that the bandwidth requests arrive at the CCN randomly in a sequential order. Upon arrival of each request, the CCN needs to instantaneously decide whether to accept this request and to which primary band this request is assigned if accepted, without a priori knowledge about the future requests. The online version of the knapsack problems have been widely investigated in the literature [26]–[29]. The common strategy is to predict the future requests, e.g., the arrival times and the amount of demanded bandwidth, in a statistical model by sampling the observations on the history. For example, [29] introduces two *black-boxes* to handle the stochastic arrivals of reservation requests for hotel rooms. One black-box is the sub-optimal approximation module and the other is the sampling module which relies on the observations of the past arrivals. Our MMKP framework can be adapted to handle the online requests in an analogous fashion. For more discussions on stochastic online combinatorial optimization, refer to [30].

### C. Fairness Issue

In MMKP, our objective is to maximize the overall throughput of the cognitive radio network while the fairness among cognitive users are not discussed. High throughput and sheer fairness are usually inherently in conflict. Our MMKP framework can incorporate the fairness concern by inclining to accept the requests from those “less-selected-in-the-past” cognitive users whenever possible. Therefore, a tradeoff between the overall throughput and fairness can be achieved.

## VI. Conclusions

In this paper, the centralized spectrum allocation of cognitive radio networks is formulated as a multiple multidimensional knapsack problem (MMKP). For the exact solution of this problem, a decomposition-based approach is presented to calculate the upper bound which is essential for the branch-and-bound algorithm. In addition, we propose a heuristic algorithm for MMKP with a performance guarantee. Our framework can be extended to many other scenarios straightforwardly.

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