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Stability of Discrete Time Jump Linear Systems*

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Abstract

In this paper, we study almost sure and moment stability of discrete time jump linear systems with a finite state iid or Markovian jump form process. A necessary and sufficient condition for almost sure stability for a special class of jump linear systems and a sufficient condition for almost sure stability of general jump linear systems are presented. We also prove that the concepts of $\delta-$moment stability, exponential $\delta-$moment stability and stochastic $\delta-$moment stability are equivalent, each of which implies almost sure stability. Also for sufficiently small δ, almost sure stability and $\delta-$moment stability are equivalent and the $\delta-$moment stability region monotonically converges to the almost sure stability region as $\delta \downarrow 0^+$. This generalizes the result in [18]. The top Lyapunov exponent and $\delta-$moment Lyapunov exponent are also studied and some relationships between them are presented.

Key words: jump linear systems, almost sure stability, $\delta-$moment stability, Lyapunov exponent

1 Introduction

Consider the discrete-time jump linear system in the form

$$x_{k+1} = H(\sigma_k)\, x_k \tag{1.1}$$

or the continuous-time jump linear system

$$\dot{x}_t = H(\xi_t)\, x_t \tag{1.2}$$

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where σ_k is a finite state independent identically distributed (iid) random process or a time homogeneous discrete time Markov chain, and $\{\xi_t\}$ is a finite state time homogeneous continuous time Markov chain. The models (1.1) and (1.2) can be used to analyze the closed-loop stability of control systems with communication delays ([1],[2]) or the stability of control systems subject to abrupt phenomena such as component and interconnection failure ([3]). The stability analysis of (1.1) or (1.2) is therefore very important in the design and analysis of such control systems. Stability analysis of systems of this type can be traced back to the work of Rosenbloom ([4]), who was interested in the moment stability properties. Bellman ([5]) was the first to study the moment stability of (1.1) with an iid form process using the Kronnecker matrix product. Bergen ([6]) used a similar idea to study the moment stability properties of the continuous time systems (1.2) with piecewise constant form process $\{\sigma_t\}$. Later, Bhuracha ([7]) used Bellman's idea developed in [5] to generalize Bergen's results and studied both asymptotic stability of the mean and exponential stability of the mean. Darkhovskii and Leibovich ([8]) investigated second moment stability of system (1.2) where the time intervals between jumps are iid and the modes of the system are governed by a finite state Markov chain with a stationary probability transition matrix. They obtained necessary and sufficient conditions for second moment stability in terms of the Kronnecker matrix product for the second moment stability, which is an extension of Bhuracha's result.

There is an alternative approach to the study of stochastic stability. Kats and Krasovskii ([9]) and Bertram and Sarachik ([10]) used a stochastic version of Lyapunov's second method to study almost sure stability and moment stability. Unfortunately, constructing an appropriate Lyapunov function is difficult in general, this is a common disadvantage of Lyapunov's second method. Also, in many cases, the criteria obtained from this method are similar to moment stability criteria, which are often too conservative. For certain classes of systems, such as (1.1) or (1.2), it is possible to obtain testable stability conditions. Ji et al. ([11]) and Feng et al. ([12],[18]) used Lyapunov's second method to study the stability of (1.1) or (1.2) where $\{\sigma_k\}$ or $\{\xi_t\}$ is a finite state Markov chain. Necessary and sufficient conditions are obtained for second moment stability of both discrete time (1.1) and continuous time (1.2) jump linear systems.

As Kozin ([13]) pointed out, moment stability implies almost sure stability under fairly general conditions, but the converse is not true. In practical applications, almost sure stability is more than often the more desirable property because we can only observe the sample path behavior of the system and the moment stability criteria are sometimes too conservative to be practically useful.

Although Lyapunov exponent techniques may provide necessary and sufficient conditions for almost sure stability ([12],[14],[15],[18],[26]-[30]), it is very difficult to compute the top Lyapunov exponent or to obtain good estimates of the top Lyapunov exponent for almost sure stability. As a result, testable conditions are difficult to obtain from this theory.

Arnold et al. ([21]) studied the relationship between the top Lyapunov exponent and the $\delta-$moment top Lyapunov exponent for a diffusion process. Using a similar idea, Leizarowitz ([25]) obtained similar results for (1.2). A general conclusion was that $\delta-$moment stability implies almost sure stability. Thus sufficient conditions for almost sure stability can be obtained through $\delta-$moment stability, which is one of the motivations for the study of $\delta-$moment stability. There are many definitions for moment stability: $\delta-$moment stability, exponential $\delta-$moment stability and stochastic $\delta-$ moment stability. Ji et al. ([11]) proved that all second moment ($\delta = 2$) stability concepts are equivalent for the system (1.1). Feng et al. ([18]) showed that all the second moment stability concepts are equivalent for the system (1.2), and also proved that for a one dimensional system of type (1.2), the region for $\delta-$moment stability is monotonically converging to the region for almost sure stability as $\delta \downarrow 0^+$. This is tantamount to concluding that almost sure stability is equivalent to $\delta-$moment stability for sufficiently small δ. This is a significant result because the study of almost sure stability can be reduced to the study of $\delta-$moment stability.

This paper is a continuation of our research work reported in [19]. In [19], some testable sufficient conditions were obtained for moment stability and for almost sure stability for systems of type (1.1), especially for the iid case, although most of the results also hold for the Markovian case. For one dimensional systems, the conditions obtained in [19] are also necessary. Also in [19], we studied the almost sure stability of system (1.1), when the individual system matrices commute, and obtained some necessary and sufficient conditions for almost sure stability. For the one dimensional case, the relationship between ($\delta-$)moment stability and almost sure stability was obtained.

In this paper, a necessary and sufficient condition for almost sure stability of a class of systems of type (1.1) are obtained, this completely solves the almost sure stability problem for these systems with commuting structures, from which we prove our conjecture in [19]. A sufficient condition for almost sure stability and $\delta-$moment stability for the general class of systems (1.1) is also presented, this generalizes our previous results ([19]). Next we study the properties of $\delta-$moment stability for (1.1) with arbitrary dimension. We show that $\delta-$moment stability, exponential $\delta-$moment stability and stochastic $\delta-$moment stability are equivalent and the region for $\delta-$moment stability is monotonically converging to the region for almost sure stability. This is a generalization of Feng et al.'s result ([18]) to discrete time systems. As a special case, a simpler proof of Ji et al.'s result is given. We also study the top $\delta-$moment Lyapunov exponent and the top Lyapunov exponent for the system (1.1), and develop some properties of the top $\delta-$moment Lyapunov exponent and the relationship between the top $\delta-$moment Lyapunov exponent and the top Lyapunov exponent. Finally, some illustrative examples are given.

2 Definitions

Throughout the rest of the paper, we study the discrete time jump linear system given by

$$x_{k+1} = H(\sigma_k)x_k, \quad k \geq 0; \tag{2.1}$$

where σ_k is either a finite state independent identically distributed process with state space $\underline{N} = \{1, 2, \ldots, N\}$ with probability distribution $P\{\sigma_0 = j\} = p_j$ for $j \in \underline{N}$ or a finite state and time homogeneous Markov chain with state space $\underline{N}$, transition probability matrix $P = (p_{ij})_{N \times N}$ and initial distribution $p = (p_1, \ldots, p_N)$ (Certainly, the iid process is a particular Markov chain where each row of P is the same as the initial distribution p). For simplicity, assume that the initial state $x_0 \in \mathcal{R}^n$ is a (nonrandom) constant vector. Let $(\Omega, \mathcal{F}, P)$ denote the underlying probability space and let Ξ be the collection of all probability distributions on $\underline{N}$. Let $e_i \in \Xi$ be the initial distribution concentrated at the i^{th} state, i.e., given by $P\{\sigma_0 = i\} = 1$. Sometimes, we need to signify that some properties are dependent on the choice of the initial distribution of the Markovian form process $\{\sigma_k\}$. If so, for each $\xi \in \Xi$, let P_ξ denote the probability measure for the Markov chain $\{\sigma_k\}$ induced by the initial distribution ξ and E_ξ the expectation with respect to P_ξ. Let $\pi = (\pi_1, \ldots, \pi_N)$ be the unique invariant probability distribution for the Markov chain $\{\sigma_k\}$, if the chain possesses a single ergodic (indecomposable) class. For a matrix $C = (c_{ij})$, let $|C| = (|c_{ij}|)$. Some definitions of different stability concepts for jump linear systems are presented next.

Definition 2.1 *Let Φ be a subset of Ξ. The jump linear system (2.1) with a Markovian form process $\{\sigma_k\}$ as specified above is said to be*

(I). (asymptotically) $\delta-$moment stable with respect to (w.r.t.) Φ, if for any $x_0 \in R^n$ and any initial probability distribution $\psi \in \Phi$ of σ_k,

$$\lim_{k \to \infty} E\left\{\|x_k(x_0, \omega)\|^\delta\right\} = 0,$$

where $x_k(x_0, \omega)$ is a sample solution of (2.1) initial from $x_0 \in R^n$. If $\delta = 2$, we say that the system (2.1) is asymptotically mean square stable w.r.t. Φ. If $\delta = 1$, we say that the system (2.1) is asymptotically mean stable w.r.t. Φ. If $\Phi = \Xi$, we simply say the system

(2.1) is asymptotically δ-moment stable. Similar statements apply to the following definitions.

(II). exponentially $\delta-$moment stable w.r.t. Φ, *if for any $x_0 \in R^n$ and any initial distribution $\psi \in \Phi$ of σ_k, there exist constants $\alpha, \beta > 0$ independent of x_0 and ψ such that*

$$E\left\{\|x_k(x_0,\omega)\|^\delta\right\} \le \alpha \|x_0\|^\delta e^{-\beta k}, \quad \forall k \ge 0.$$

(III). stochastically $\delta-$moment stable w.r.t. Φ, *if for any $x_0 \in R^n$ and any initial distribution $\psi \in \Phi$ of σ_k,*

$$\sum_{k=0}^{\infty} E\left\{\|x_k(x_0,\omega)\|^\delta\right\} < +\infty.$$

(IV). almost surely (asymptotically) stable w.r.t. Φ, *if for any $x_0 \in R^n$ and any initial distribution $\psi \in \Xi$ of σ_k,*

$$P\left\{\lim_{k\to\infty} \|x_k(x_0,\omega)\| = 0\right\} = 1.$$

(V). mean value stable w.r.t. Φ, *if for any $x_0 \in R^n$ and any initial distribution $\psi \in \Phi$ of σ_k,*

$$\lim_{k\to\infty} E\left\{x_k(x_0,\omega)\right\} = 0.$$

In the case when $\{\sigma_k\}$ is actually an iid process with distribution $p = (p_1, \cdots, p_N)$, all the above definitions hold with Φ being the singleton set $\Xi = \Phi = \{p\}$.

The above definitions are consistent with those given in [11] and [18], and we want to remind the reader of the dependence on the initial probability distribution of the form process $\{\sigma_k\}$ for the Markovian case. The "state" for the jump linear system is the joint process (x_k, σ_k), even though the initial distribution of the form process may not be known. Thus, it is reasonable that the stability properties as given are independent of the initial distributions. Of course, for a Markov chain with a single ergodic class, the almost sure (sample) stability only depends on the probability measure P_π with the initial distribution π. Then, if the system is P_π-almost surely stable, then it is also almost surely stable (or P_ξ-almost surely stable for any $\xi \in \Xi$). However, this may not be the case for δ-moment stability. The following simple example illustrates this point and justifies the practical importance of having the stability definitions independent of the initial distribution.

Example 2.2 Consider the scalar system (2.1) with $H(1) = h_1 > 0$ and $H(2) = h_2$ with $0 < h_2 < 1$. The form process $\{\sigma_k\}$ has a transition matrix

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} = \begin{pmatrix} 0.5 & 0.5 \\ 0 & 1 \end{pmatrix}$$

and initial distribution $\xi = (\xi_1, \xi_2)$. Clearly, the unique invariant distribution is given by $\pi = e_2 = (0,1)$ and the system is P_ξ-almost surely stable, regardless of ξ. However, for any $\delta > 0$ and $x_0 \neq 0$, we have

$$\begin{aligned} E_\xi |x_n(\omega, x_0)|^\delta &= E_\xi |H(\sigma_{n-1}) \dots H(\sigma_0) x_0|^\delta \\ &= \sum_{i=1}^{2} \xi_i E_{e_i} |H(\sigma_{n-1}) \dots H(\sigma_0) x_0|^\delta. \end{aligned} \tag{2.2}$$

Also, we have

$$\begin{aligned} & E_{e_1} |H(\sigma_{n-1}) \dots H(\sigma_0) x_0|^\delta \\ &= \sum_{i_1, \dots, i_{n-1}} p_{1 i_1} p_{i_1 i_2} \dots p_{i_{n-2} i_{n-1}} |H(i_{n-1})|^\delta \dots |H(i_1)|^\delta |H(1)|^\delta |x_0|^\delta \\ &\geq (\frac{1}{2})^n h_1^{n\delta} |x_0|^\delta = (\frac{1}{2} h_1^\delta)^n |x_0|^\delta, \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} & E_{e_2} |H(\sigma_{n-1}) \dots H(\sigma_0) x_0|^\delta \\ &= \sum_{i_1, \dots, i_n} p_{2 i_1} p_{i_1 i_2} \dots p_{i_{n-2} i_{n-1}} |H(i_{n-1})|^\delta \dots |H(i_1)|^\delta |H(2)|^\delta |x_0|^\delta \\ &= h_2^{n\delta} |x_0|^\delta = (h_2^\delta)^n |x_0|^\delta \longrightarrow 0, \quad n \to +\infty. \end{aligned} \tag{2.4}$$

From (2.2), (2.3) and (2.4), we see that

$$\lim_{n \to +\infty} E_\pi |x_n(\omega, x_0)|^\delta = \lim_{n \to +\infty} E_{e_2} |x_n(\omega, x_0)|^\delta = 0.$$

However, for any $\xi = (\xi_1, \xi_2)$ with $\xi_1 > 0$, as long as $h_1^\delta > 2$, we have

$$\lim_{n \to +\infty} E_\xi |x_n(\omega, x_0)|^\delta \geq \lim_{n \to +\infty} \xi_1 E_{e_1} |x_n(\omega, x_0)|^\delta = +\infty.$$

In this case, the system is "δ-moment stable", if $\xi = \pi$, i.e., if the chain $\{\sigma_k\}$ is stationary, and the system is not "δ-moment stable" for any other initial distribution ξ. Therefore, δ-moment stability with respect to $\Phi = \{\pi\}$ is not a good criteria to be used in practice because a small perturbation of ξ from π will make the system unstable. The δ-moment stability definition should therefore be "independent" of the initial distribution as given in Definition 2.1.

In the above example, the form process has a single ergodic class $\{2\}$ as well as a transient state, namely 1. If the form process is irreducible, i.e., satisfies the property that each pair of states communicates, or that the unique invariant distribution π is strictly positive, then, the definitions in 2.1 are equivalent to the usual stability definitions for a system with a stationary form process. This result is formalized next.

Lemma 2.3 *For system (2.1) with a finite state and time homogeneous form process, if the chain is irreducible (or indecomposable) with a unique invariant distribution π, then the system is stable in any of the above senses if and only if the system is stable in the same sense with respect to $\Phi = \{\pi\}$.*

Proof: The proof of necessity is trivial. For sufficiency, notice that since $\pi > 0$, it is easy to see that $P_\xi << P_\pi$ (P_ξ is absolutely continuous with respect to P_π) for any $\xi \in \Xi$. Thus, P_π-almost sure stability implies P_ξ-almost sure stability. For moment properties, say, δ-moment stability, notice that for any $\xi = (\xi_1, \ldots, \xi_N)$,

$$E_\xi \|x_k(\omega, x_0)\|^\delta = \sum_{i=1}^{N} \xi_i E_{e_i} \|x_k(\omega, x_0)\|^\delta.$$

Since $\pi = (\pi_1, \ldots, \pi_N) > 0$, $\lim_{k\to+\infty} E_\pi \|x_k(\omega, x_0)\|^\delta = 0$ implies that $\lim_{k\to+\infty} E_{e_i} \|x_k(\omega, x_0)\|^\delta = 0$ for all $i \in \underline{N}$. This implies that

$$\lim_{k\to+\infty} E_\xi \|x_k(\omega, x_0)\|^\delta = 0$$

for all $\xi \in \Xi$. □

We conclude that if we are dealing with an irreducible Markov chain form process, then it is only necessary to study stability with respect to $\Phi = \{\pi\}$.

3 Almost Sure Stability

In [19], we developed many criteria for the stability of the jump linear system (2.1) predominantly for the case when $\{\sigma_k\}$ is an independent identically distributed finite state processes. Most of these results can be generalized to the case when $\{\sigma_k\}$ is a finite state Markov process. For systems of type (2.1) where the state matrices commute, we gave some necessary and sufficient conditions for certain special cases. Here, we prove a more general result which gives necessary and sufficient conditions for almost sure stability of (2.1) with an iid form process and commuting state matrices.

Theorem 3.1 *For the system (2.1) with $\{\sigma_k\}$ a finite state independent identically distributed random process with $\{p_1, p_2, \ldots, p_N\}$ the common probability distribution, assume that each of the $n \times n$ matrices $H(1), H(2), \ldots, H(N)$ can be simultaneously transformed by a similarity transformation to upper triangular form with the diagonal elements $\lambda_{1,j}, \lambda_{2,j}, \ldots, \lambda_{n,j}$ for $j \in \underline{N}$. Then, a necessary and sufficient condition for almost sure stability is*

$$|\lambda_{i,1}^{p_1} \lambda_{i,2}^{p_2} \cdots \lambda_{i,N}^{p_N}| < 1 \qquad \forall i = 1, 2, \ldots, n. \tag{3.1}$$

In particular, if $H(1), H(2), \ldots, H(N)$ pairwise commute, then there exists a unitary matrix T such that

$$T^{-1}H(j)T = \begin{pmatrix} \lambda_{1,j} & * & \cdots & * \\ & \lambda_{2,j} & \cdots & * \\ & & \ddots & * \\ & & & \lambda_{n,j} \end{pmatrix} \qquad \forall j = 1, 2, \ldots, N,$$

and (3.1) is a necessary and sufficient condition for almost sure stability.

To prove this theorem, we need the following lemma, which infers stability of (2.1) from the stability of a dominating system.

Lemma 3.2 *For (2.1) with an iid form process, let $(p_1, \ldots, p_N)$ be the common probability distribution of $\{\sigma_k\}$.*

(i) Let $H(r) = (h_{ij}(r))$ and $\bar{H}(r) = (\bar{h}_{ij}(r))$ with $|h_{ij}(r)| \leq \bar{h}_{ij}(r)$ for all $r \in \underline{N}$. If the dominant system defined by

$$x_{k+1} = \bar{H}(\sigma_k)\, x_k, \quad x(0) = x_0 \tag{3.2}$$

is almost surely stable, then the system (2.1) is almost surely stable.

(ii) Let $|H(r)| = (|h_{ij}(r)|)$ for $r = 1, 2, \ldots, N$. If

$$E|H(\sigma_0)| = p_1|H(1)| + p_2|H(2)| + \cdots + p_N|H(N)|$$

is a stable matrix, then the system (2.1) is almost surely stable. In particular, if $H(1), H(2), \ldots, H(N)$ are nonnegative matrices, then the system (2.1) is almost surely stable if

$$EH(\sigma_0) = p_1H(1) + p_2H(2) + \cdots + p_NH(N)$$

is a stable matrix.

Proof: To show (i), it is sufficient to prove that if $\bar{H}(\sigma_k)\bar{H}(\sigma_{k-1}) \cdots \bar{H}(\sigma_0)$ converges to zero almost surely, then $H(\sigma_k)H(\sigma_{k-1}) \cdots H(\sigma_1)$ also converges to zero almost surely. However, $|H(\sigma_k)H(\sigma_{k-1}) \ldots H(\sigma_0)| \leq \bar{H}(\sigma_k)\bar{H}(\sigma_{k-1}) \cdots \bar{H}(\sigma_0)$ and the result follows directly.

For (ii), let $G_k = |H(\sigma_k)| \cdots |H(\sigma_0)|$. If $E|H(\sigma_0)|$ is stable, then by the iid property, $EG_k = (E|H(\sigma_0)|)^{k+1}$ exponentially converges to zero. Thus, with $G_k = (g_{ij}(k))$, there exists an $M > 0$ and $0 < r < 1$ so that $Eg_{ij}(k) \le Mr^k$. Let $g_{ij} = \overline{\lim}_{k\to\infty} g_{ij}(k)$, since G_k is nonnegative matrix, $g_{ij}(k) \ge 0$. Hence, $g_{ij} \ge 0$. For any positive $c > 0$, by Chebyshev's inequality,

$$P\{g_{ij} > c\} = P\{\cap_{m=1}^{\infty} \cup_{n=m}^{\infty} \{g_{ij}(n) > c\}\} \le P\{\cup_{n=m}^{\infty}\{g_{ij}(n) > c\}\}$$
$$\le \sum_{n=m}^{\infty} P\{g_{ij}(n) > c\} \le \frac{1}{c}\sum_{n=m}^{\infty} Eg_{ij}(n) = \frac{1}{c}\sum_{n=m}^{\infty} Mr^n,$$

and letting m go to infinity, we obtain that $P\{g_{ij} > c\} = 0$ for any $c > 0$. From this, we can obtain that $P\{g_{ij} = 0\} = 1$. Hence, $\lim_{k\to\infty} g_{ij}(k) = 0$ almost surely. This means that $x_{k+1} = |H(\sigma_k)| x_k$ is almost surely stable. From (i), we conclude that (2.1) is almost surely stable. In the case when $H(j)$ is nonnegative for all $j \in \underline{N}$, we have $\bar{H}(j) = H(j)$ and the result follows. □

Remark: This lemma gives very simple criteria for almost sure stability of the system (2.1) with an iid form process. Notice that $EH(\sigma_0)$ is stable if and only if (2.1) is mean value stable. Thus, from (ii) we see that for (2.1) with nonnegative form matrices, mean value stability implies (but is not necessarily equivalent to) almost sure stability, which is not true in general (see the examples which are given later).

Proof of Theorem 3.1: Without loss of generality, we can assume that $H(1), H(2), \ldots, H(N)$ are all upper triangular matrices. Let b be the upper bound of the absolute values of the off-diagonal elements of $H(1), \ldots, H(N)$, and with a slight abuse of notation, let $\lambda_{1,j}, \lambda_{2,j}, \ldots, \lambda_{n,j}$ be the absolute values of the diagonal elements of $H(j)$ for $j \in \underline{N}$. To prove the sufficiency, from Lemma 3.2, it suffices to show that if

$$\lambda_{i,1}^{p_1} \lambda_{i,2}^{p_2} \cdots \lambda_{i,N}^{p_N} < 1 \quad \forall i = 1, 2, \ldots, n$$

then the dominating system defined by

$$x_{k+1} = \bar{H}(\sigma_k)\, x_k, \tag{3.3}$$

with

$$\bar{H}(j) = \begin{pmatrix} \lambda_{1,j} & b & \ldots & b \\ & \lambda_{2,j} & \ldots & b \\ & & \ddots & \vdots \\ & & & \lambda_{n,j} \end{pmatrix}, \quad j = 1, 2, \ldots, N,$$

is almost surely stable. Let

$$G_k = \bar{H}(\sigma_k)\cdots\bar{H}(\sigma_0) = \begin{pmatrix} g_{11}^{(k)} & g_{12}^{(k)} & \cdots & g_{1n}^{(k)} \\ & g_{22}^{(k)} & \cdots & g_{2n}^{(k)} \\ & & \ddots & \vdots \\ & & & g_{nn}^{(k)} \end{pmatrix}.$$

The almost sure stability of (3.3) is equivalent to the almost sure convergence of G_k to the zero matrix. Let $\lambda_1^{(\sigma_k)}, \lambda_2^{(\sigma_k)}, \ldots, \lambda_n^{(\sigma_k)}$ be the diagonal elements of $\bar{H}(\sigma_k)$. From $G_k = H(\sigma_k)\, G_{k-1}$ and the triangular form of matrices $H(\sigma_k)$, we obtain the recursive equations for the last column of G_k:

$$\begin{aligned} g_{1n}^{(k)} &= \lambda_1^{(\sigma_k)} g_{1n}^{(k-1)} + b(g_{2n}^{(k-1)} + \ldots + g_{nn}^{(k-1)}) \\ g_{2n}^{(k)} &= \lambda_2^{(\sigma_k)} g_{2n}^{(k-1)} + b(g_{3n}^{(k-1)} + \ldots + g_{nn}^{(k-1)}) \\ &\vdots \\ g_{(n-1)n}^{(k)} &= \lambda_{n-1}^{(\sigma_k)} g_{(n-1)n}^{(k-1)} + b g_{nn}^{(k-1)} \\ g_{nn}^{(k)} &= \lambda_n^{(\sigma_k)} g_{nn}^{(k-1)}. \end{aligned} \tag{3.4}$$

To proceed, we use induction on the dimension n. If $n = 1$, it is easy to show that $G_k \to 0$ almost surely and the result is valid. Suppose that it is true for $n-1$. Because of the triangular structure of G_j and the induction hypothesis, it is sufficient to show that when $\dim G_k = n$, the elements of the last column of the matrix G_k are converging to zero almost surely. From the last equation of (3.4), we have

$$\begin{aligned} g_{nn}^{(k)} &= \lambda_n^{(\sigma_k)} \lambda_n^{(\sigma_{k-1})} \cdots \lambda_n^{(\sigma_1)} g_{nn}^{(0)} \\ &= \left(\lambda_{n,1}^{\frac{1}{k}\sum_{i=1}^k I_1(\sigma_i)} \cdots \lambda_{n,N}^{\frac{1}{k}\sum_{i=1}^k I_N(\sigma_i)} \right)^k g_{nn}^{(0)} \end{aligned} \tag{3.5}$$

where $I_r(j) = \delta_{rj}$ is the Kronecker delta (indicator) function. By hypothesis $\lambda_{n,1}^{p_1} \lambda_{n,2}^{p_2} \cdots \lambda_{n,N}^{p_N} < 1$ and the Law of Large Numbers, there exists a $0 < \rho_n < 1$ and $M_n(\omega) > 0$, such that for all k,

$$|g_{nn}^{(k)}| \le M_n(\omega)\rho_n^k.$$

Now, we use an induction argument on the index j of the elements $g_{jn}^{(k)}$ to show that for each $1 \le i \le n$, there exists $M_i(\omega)$ and $0 < \rho_i < 1$ with $M_i(\omega)$ a polynomial of finite degree in the variable k so that

$$|g_{in}^{(k)}| \le M_i(\omega)\rho_i^k. \tag{3.6}$$

From this, we can conclude that $g_{in}^{(k)}$ goes to zero almost surely for each $1 \leq i \leq n$. We proceed as follows: Suppose that for some $1 < j < n$, there exists $M_j(\omega), \ldots, M_n(\omega) > 0$ and $\rho_j, \ldots, \rho_n < 1$, such that

$$|g_{in}^{(k)}| \leq M_i(\omega)\rho_i^k \quad \forall i = j, j+1, \ldots, n.$$

We show that (3.6) holds for $i = j-1$. From the $(j-1)^{\text{th}}$ equation of (3.4), we have

$$\begin{aligned} g_{(j-1)n}^{(k)} =& \lambda_{j-1}^{(\sigma_k)}\lambda_{j-1}^{(\sigma_{k-1})}\cdots\lambda_{j-1}^{(\sigma_1)}g_{(j-1)n}^{(0)} + \\ &+ \sum_{l=1}^{k-1} b\lambda_{j-1}^{(\sigma_k)}\cdots\lambda_{j-1}^{(\sigma_{l+1})}(g_{jn}^{(l-1)} + \cdots + g_{nn}^{(l-1)}) + \\ &+ b(g_{jn}^{(k-1)} + \cdots + g_{nn}^{(k-1)}). \end{aligned} \tag{3.7}$$

We see that the last term in (3.7) almost surely converges to zero at an exponential rate because of the induction hypothesis. Actually, let $\overline{M}_{j-1}(\omega) = \max_{j \leq i \leq n} M_i(\omega)$, and $\overline{\rho}_{j-1} = \max_{j \leq i \leq n} \rho_i$. We have

$$|b(g_{jn}^{(k-1)} + \cdots + g_{nn}^{(k-1)})| \leq nb\overline{M}_{j-1}(\omega)\overline{\rho}_{j-1}^{k-1}. \tag{3.8}$$

By the Law of Large Numbers, similar to (3.5), we also have for the first term in (3.7) that there exist $\widetilde{M}_{j-1}(\omega)$ and $0 < \widetilde{\rho}_{j-1} < 1$ such that

$$|\lambda_{j-1}^{(\sigma_k)}\lambda_{j-1}^{(\sigma_{k-1})}\cdots\lambda_{j-1}^{(\sigma_1)}g_{(j-1)n}^{(0)}| \leq \widetilde{M}_{j-1}(\omega)\widetilde{\rho}_{j-1}^{k}. \tag{3.9}$$

For the second term in (3.7), consider that

$$\lambda_{j-1}^{(\sigma_k)}\cdots\lambda_{j-1}^{(\sigma_{l+1})} = \left(\lambda_{j-1,1}^{\frac{1}{k-l}\sum_{i=l+1}^{k-1} I_1(\sigma_i)} \cdots \lambda_{j-1,N}^{\frac{1}{k-l}\sum_{i=l+1}^{k-1} I_N(\sigma_i)}\right)^{(k-l)}.$$

Again, by the Law of Large Numbers and the hypothesis of the theorem, there exist $\hat{M}_{j-1}(\omega) > 0$ and $0 < \hat{\rho}_{j-1} < 1$ such that

$$\lambda_{j-1}^{(\sigma_k)}\cdots\lambda_{j-1}^{(\sigma_{l+1})} \leq \hat{M}_{j-1}(\omega)\hat{\rho}_{j-1}^{k-l} \quad \text{a.s..}$$

Let

$$\rho_{j-1} = \max\{\overline{\rho}_{j-1}, \widetilde{\rho}_{j-1}, \hat{\rho}_{j-1}\}$$

and let

$$M'_{j-1}(\omega) = \max\{\overline{M}_{j-1}(\omega), \hat{M}_{j-1}(\omega)\}.$$

Then, it follows that the second term in (3.7) satisfies the inequality

$$
\begin{aligned}
&|\sum_{l=1}^{k-1} b\lambda_{j-1}^{(\sigma_k)}\cdots\lambda_{j-1}^{(\sigma_l+1)}(g_{jn}^{(l-1)}+\cdots+g_{nn}^{(l-1)})| \\
&\quad\le \sum_{l=1}^{k-1} b(\hat{M}_{j-1}(\omega)\hat{\rho}_{j-1}^{k-l})(M_j(\omega)\rho_2^l+\cdots+M_n(\omega)\rho_n^l) \qquad (3.10)\\
&\quad\le nbM'^2_{j-1}(\omega)\sum_{l=1}^{k-1}\rho_{j-1}^k \le nbM'^2_j(\omega)k\rho_{j-1}^k \text{ a.s.}
\end{aligned}
$$

Combining (3.8), (3.9) and (3.10), we conclude that there exists $M_{j-1}(\omega)$ (which is a finite degree polynomial in k) such that

$$|g_{(j-1)n}^{(k)}| \le M_{j-1}(\omega)\rho_{j-1}^k \overset{k\to\infty}{\longrightarrow} 0 \quad \text{a.s.}$$

Induction on the dimension n guarantees that the system (3.1) and hence the system (2.1) is almost surely stable for arbitrary dimension, if the conditions of the theorem are satisfied. This completes the proof of sufficiency.

Necessity can be proved by observing that if the diagonal elements of the product of matrices G_k converges to zero as k goes to infinity, then the conditions of the theorem are satisfied (see [19] for a similar proof). □

Remark:

1. We may prove Theorem 3.1 directly from the system (2.1) with an iid form process using the result in [19] for the one dimensional case, the procedure will be the same as above. Notice that from the above proof, under the condition of Theorem 3.1, the system is actually almost surely exponentially stable.
2. It is evident that Theorem 3.1 is also valid for systems with a finite state and ergodic Markov chain. In this case, p_i should be replaced by π_i.

Lyapunov's second method is a very important technique for the study of stability of dynamical systems. Hasminskii [20] used a stochastic version of this theory to study the stability of stochastic systems. Using this method, we can obtain improved criteria for almost sure stability when compared to the results we obtained previously in [19]. In what follows, we use $\|\cdot\|$ to denote any vector norm on $\mathbf{R}^n$, $\|x\|$ for $x\in\mathbf{R}^n$ and for any induced matrix norm on $\mathbf{R}^{n\times n}$, $\|A\|$ for $A\in\mathbf{R}^{n\times n}$. We use $\|\cdot\|_2$ to denote the 2−norm, i.e., the Euclidean norm.

Lemma 3.3: (Hasminskii [20], pp. 214-215) *Let $\{A_n\}$ be an iid matrix sequence, a sufficient condition for*

$$E\|A_nA_{n-1}\cdots A_1\|^\delta \overset{n\to\infty}{\longrightarrow} 0$$

is that there exists a positive definite function $f(x)$, homogeneous of degree δ, such that the function $Ef(A_1x) - f(x)$ is negative definite of degree δ, i.e., there exists $K > 0$ such that $Ef(A_1x) - f(x) \leq -K\|x\|^\delta$.

From this we obtain:

Theorem 3.4: *Let $\{\sigma_k\}$ be a finite state iid sequence with a common probability distribution $\{p_1, p_2, \dots, p_N\}$. Define*

$$\mu^\delta_{max} = \max_{\|x\|=1} \{p_1\|H(1)x\|^\delta + p_2\|H(2)x\|^\delta + \cdots + p_N\|H(N)x\|^\delta\}$$
$$\mu^\delta_{min} = \min_{\|x\|=1} \{p_1\|H(1)x\|^\delta + p_2\|H(2)x\|^\delta + \cdots + p_N\|H(N)x\|^\delta\}.$$

Then, the system (1.1) is $\delta-$moment stable if $\mu^\delta_{max} < 1$, and the system (1.1) is $\delta-$moment unstable if $\mu^\delta_{min} \geq 1$.

Proof: Let $f(x) = \|x\|^\delta$, then

$$
\begin{aligned}
& Ef(H(\sigma_1)x) - f(x) \\
&= p_1\|H(1)x\|^\delta + p_2\|H(2)x\|^\delta + \cdots + p_N\|H(N)x\|^\delta - \|x\|^\delta \\
&= \|x\|^\delta \left(p_1\|H(1)\frac{x}{\|x\|}\|^\delta + \cdots + p_N\|H(N)\frac{x}{\|x\|}\|^\delta - 1\right) \\
&\leq \|x\|^\delta(\mu^\delta_{max} - 1).
\end{aligned}
$$

Then from Lemma 3.3, if $\mu^\delta_{max} < 1$, the system (2.1) is $\delta-$moment stable. In a similar fashion, we can prove that if $\mu^\delta_{min} \geq 1$, the system is $\delta-$moment unstable. □

We will study the relationship between almost sure and δ-moment stability in the next section. It will be shown that δ-moment stability implies almost sure stability. From this fact, together with the above theorem, we obtain the following criterion for the almost sure stability of (2.1).

Theorem 3.5: *Let $\{\sigma_k\}$ be a finite state independent identically distributed sequence with common distribution $\{p_1, p_2, \dots, p_N\}$, then the system (2.1) is almost surely stable if $\sigma_{max} < 1$, where*

$$\sigma_{max} = \max_{\|x\|=1} \{\|H(1)x\|^{p_1} \cdots \|H(N)x\|^{p_N}\}.$$

Proof: If $\sigma_{max} < 1$, we claim that there exists a $\delta > 0$ such that $\mu^\delta_{max} < 1$. Otherwise, for any k, $\delta = \frac{1}{k}$, there exists x_k satisfying $\|x_k\| = 1$, such that

$$p_1\|H(1)x_k\|^{1/k} + \cdots + p_N\|H(N)x_k\|^{1/k} \geq 1. \qquad (3.11)$$

Without loss of generality (because the unit sphere is compact and $\{x_k\}$ is a sequence in the sphere), we can assume that there exists an x_0 satisfying $\|x_0\| = 1$, such that $\lim_{k\to\infty} x_k = x_0$. Hence for any j, $\lim_{k\to\infty} \|H(j)x_k\| = \|H(j)x_0\|$. It follows that for any $\varepsilon > 0$, and sufficiently large k, we have

$$\|H(j)x_k\| \leq \|H(j)x_0\| + \varepsilon \tag{3.12}$$

(3.11) and (3.12) gives

$$\begin{aligned} 1 &\leq \left(p_1\|H(1)x_k\|^{1/k} + \cdots + \|H(N)x_k\|^{1/k}\right)^k \\ &\leq \left(p_1(\|H(1)x_0\| + \varepsilon)^{1/k} + \cdots + p_N(\|H(N)x_0\| + \varepsilon)^{1/k}\right)^k. \end{aligned}$$

Letting $k \to \infty$, we obtain

$$1 \leq (\|H(1)x_0\| + \varepsilon)^{p_1} \cdots (\|H(N)x_0\| + \varepsilon)^{p_N},$$

and letting $\varepsilon \to 0$, we obtain

$$\|H(1)x_0\|^{p_1}\|H(2)x_0\|^{p_2} \cdots \|H(N)x_0\|^{p_N} \geq 1.$$

This means that $\sigma_{max} \geq 1$, which contradicts the assumption. Therefore there exists $\delta > 0$ such that $\mu^{\delta}_{max} < 1$ and from Thereom 3.4, the system is $\delta-$moment stable. It will be shown that δ-moment stability implies almost sure stability (see Corollary 4.4 in next section). This completes the proof. □

The following example illustrates that this criterion is better than those previously developed in [19].

Example 3.6:

$$H(1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ H(2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \ p_1 = p_2 = 0.5.$$

Since $\|H(1)\|_2 = \|H(2)\|_2 = 1$, we can not guarantee the almost sure stability of the system using the criteria given in terms of singular values in [19]. However,

$$\sigma^2_{max} = \max_{\|x\|_2=1} \|H(1)x\|_2\|H(2)x\|_2 = \max_{0\leq\theta\leq 2\pi} |\sin\theta\cos\theta| \leq \frac{1}{2},$$

and from Theorem 3.5 the system is almost surely stable.

4 $\delta-$Moment Stability

$\delta-$Moment Stability and its Relationship with A.S. Stability

In this section, we will study various types of $\delta-$moment stability for the jump linear system (2.1). For $\delta = 2$, Ji et al. ([11]) proved that for the system (2.1) with a finite state Markov form process, second moment stability, second moment exponential stability and second moment stochastic stability are equivalent, and all of these stability definitions imply almost sure stability. Feng et al. ([18]) obtained the same result for continuous-time jump linear systems (1.2) and further proved that for one dimensional systems, almost sure stability is equivalent to $\delta-$moment stability for sufficiently small δ. In [19], we have proved that for one dimensional discrete-time systems (2.1), almost sure stability is also equivalent to $\delta-$moment stability for sufficiently small δ. In this section, we will prove that for any positive δ, $\delta-$moment stability, exponential $\delta-$moment stability and stochastic $\delta-$moment stability are also equivalent, and all of these stability definitions imply almost sure stability. Additionally, a much simpler proof of the results for $\delta = 2$ in Ji et al.'s work ([11]) is provided. We will also prove that almost sure stability of (2.1) is equivalent to $\delta-$moment stability for sufficiently small δ, thus the conjecture given in Feng ([12]) and Feng et al. ([18]) is proved for discrete-time systems. For diffusion processes, Arnold et al. ([21]) obtained a relationship between the Lyapunov exponents and the $\delta-$moment Lyapunov exponents. We obtain a similar result for systems of the type (2.1).

We shall work on the system (2.1) with a Markov form process exclusively. The iid case is a special case, except for the following consideration: Recall that in the Markovian case, we require the stability properties are independent of the initial distributions $\xi \in \Xi$. Thus, we have to justify that when $\{\sigma_k\}$ is indeed an iid sequence with a common probability distribution $p = (p_1, \ldots, p_N)$ and is interpreted as a Markov chain with transition matrix $P = (p', p' \ldots, p')'$, the seemingly strong stability concepts (with respect to arbitrary initial distributions) for the Markovian case are coincident with the results for the iid case. This, however, directly follows from the observation that stability is an asymptotic property, and if $\{\eta_k\}_{k=0}^{+\infty}$ is a Markov chain with the transition matrix $P = (p', p', \ldots, p')'$ and any initial distribution, then $\{\eta_{k+1}\}_{k=0}^{+\infty}$ is an iid sequence with the common distribution p. We begin with the equivalence of various δ-moment stability properties:

Theorem 4.1: *For system (2.1) with a Markov form process (which is finite state and time-homogeneous), $\delta-$moment stability, exponential $\delta-$moment stability and stochastic $\delta-$moment stability are all equivalent.*

Proof: It is easy to show that exponential $\delta-$moment stability implies stochastic $\delta-$moment stability which then implies $\delta-$moment stability. Thus, to prove the equivalence, it is sufficient to show that $\delta-$moment stability implies exponential $\delta-$moment stability.

According to the definition in §2, if (2.1) is $\delta-$moment stable, then for any initial distribution $\xi \in \Xi$,

$$\lim_{n\to\infty} E_\xi \|H(\sigma_n)\cdots H(\sigma_0)\|^\delta = 0,$$

where E_ξ denotes the expectation with respect to P_ξ, the probability measure induced by ξ for $\{\sigma_k\}$.

Let $\xi_i = (p_{i1}, \ldots, p_{iN})$ be the $i-$th row of the matrix P. This defines a probability distribution for the process $\{\sigma_k\}$. Let ξ_0 be any initial distribution in Ξ with $\xi_0 = \{p_1, p_2, \ldots, p_N\}$. From the $\delta-$moment stability, we have

$$\lim_{n\to\infty} \max_{0\le i\le N} E_{\xi_i} \|H(\sigma_n)\cdots H(\sigma_0)\|^\delta = 0.$$

Then for any $0 < r < 1$ given, there exists an integer m so that

$$\max_{0\le i\le N} E_{\xi_i} \|H(\sigma_{m-1})\cdots H(\sigma_0)\|^\delta \le r. \tag{4.1}$$

Also, there exists an $M > 0$ such that for any $0 \le q < m$ and any k, we have

$$\max_{0\le i\le N} E_{\xi_i} \|H(\sigma_{k+q})\cdots H(\sigma_k)\|^\delta \le M. \tag{4.2}$$

In arriving at (4.2), we have used the time homogeneous property of $\{\sigma_k\}$.

Let $k = pm + q$, where $0 \le q < m$, then we obtain, using the time homogeneous property again,

$$
\begin{aligned}
&E_{\xi_0}\|H(\sigma_k)\cdots H(\sigma_0)\|^\delta \\
&\le E_{\xi_0}\|H(\sigma_{pm+q})\cdots H(\sigma_{pm})\|^\delta \|H(\sigma_{pm-1})\cdots \\
&H(\sigma_{(p-1)m})\|^\delta \cdots \|H(\sigma_{m-1})\cdots H(\sigma_0)\|^\delta \\
&= \sum_{i_0,\ldots,i_{pm+q}} p_{i_0}p_{i_0 i_1}\cdots p_{i_{pm+q-1}i_{pm+q}}\|H(i_{pm+q})\cdots H(i_{pm})\|^\delta \\
&\times \|H(i_{pm-1})\cdots H(i_{(p-1)m})\|^\delta \cdots \|H(i_{m-1})\cdots H(i_0)\|^\delta \\
&= \sum_{i_0,\ldots,i_{pm-1}} p_{i_0}p_{i_0 i_1}\cdots p_{i_{pm-2}i_{pm-1}}\|H(i_{pm-1})\cdots H(i_{(p-1)m})\|^\delta \\
&\times \|H(i_{m-1})\cdots H(i_0)\|^\delta \\
&\times \sum_{i_{pm},\ldots,i_{pm+q}} p_{i_{pm-1}i_{pm}}\cdots p_{i_{pm+q-1}i_{pm+q}}\|H(i_{pm+q})\cdots H(i_{pm})\|^\delta \\
&= \sum_{i_0,\ldots,i_{pm-1}} p_{i_0}p_{i_0 i_1}\cdots p_{i_{pm-2}i_{pm-1}}\|H(i_{pm-1})\cdots H(i_{(p-1)m})\|^\delta \\
&\times \|H(i_{m-1})\cdots H(i_0)\|^\delta \\
&\times E_{\xi_{i_{pm-1}}}\|H(\sigma_q)\cdots H(\sigma_0)\| \\
&\le \max_{1\le i\le N} E_{\xi_i}\|H(\sigma_q)\cdots H(\sigma_0)\|^\delta \\
&\times \sum_{i_0,\ldots,i_{pm-1}} p_{i_0}p_{i_0 i_1}\cdots p_{i_{pm-2}i_{pm-1}}\|H(i_{pm-1})\cdots H(i_{(p-1)m})\|^\delta \\
&\times \|H(i_{m-1})\cdots H(i_0)\|^\delta \\
&\le \cdots \le \max_{1\le i\le N} E_{\xi_i}\|H(\sigma_q)\cdots H(\sigma_0)\|^\delta \\
&\times \max_{1\le i\le N} E_{\xi_i}\|H(\sigma_{m-1})\cdots H(\sigma_0)\|^\delta \times \cdots \\
&\times \max_{1\le i\le N} E_{\xi_i}\|H(\sigma_{m-1})\cdots H(\sigma_0)\|^\delta \\
&= \max_{1\le i\le N} E_{\xi_i}\|H(\sigma_q)\cdots H(\sigma_0)\|^\delta \times \left[\max_{1\le i\le N} E_{\xi_i}\|H(\sigma_{m-1})\cdots H(\sigma_0)\|^\delta\right]^p \\
&\le Mr^p = M(r^{1/m})^{pm} \le M_1 r_1^k,
\end{aligned}
$$

where $M_1 = Mr^{-q/m}$ and $r_1 = r^{1/m}$. We conclude from this that (2.1) is exponentially δ−moment stable. This completes the proof. □

The above theorem establishes the equivalence of various δ-moment stability properties. This is a generalization of the result of Ji et al. [11] for second moment stability ($\delta = 2$). Next, we study the relationship between moment and almost sure stability. First of all, we prove some basic results for the general nonlinear stochastic system

$$x_{k+1} = f(\omega, x_k), \quad x_0 \in \mathcal{R}^n. \tag{4.3}$$

All the stability concepts for (4.3) are similarly defined as for the jump linear system, in a obvious way. Furthermore, we make the following definition:

Definition 4.2: *The stochastic system (4.3) is said to be weakly exponentially stable in probability, if for any $\varepsilon > 0$, there exist $M(\varepsilon) > 0$ and $0 < \gamma < 1$ (independent of ε) such that for all $k \geq 0$,*

$$P\{\|x_k(\omega, x_0)\| \geq \varepsilon\} \leq M(\varepsilon)\gamma^k, \quad \forall x_0 \in \mathcal{R}^n.$$

Proposition 4.3: *The following statements hold for the stochastic system (4.3).*

(i) If (4.3) is exponentially δ-moment stable, then it is stochastically δ-moment stable, which in turn implies that (4.3) is almost surely stable.

(ii) If (4.3) is exponentially δ-moment stable, then, it is weakly exponentially stable in probability. Furthermore, suppose that there exists $N > 0$ such that $\|x_k(\omega, x_0)\| \leq \|x_0\| N^k$ almost surely for all x_0 and k. Then, (4.3) is weakly exponentially stable in probability implies that (4.3) is δ'-moment stability for some $\delta' > 0$.

Proof: If the system is exponentially $\delta-$moment stable, then there exists $M > 0$ and $0 < \gamma < 1$, such that $E\|x_k\|^\delta \leq M\gamma^k$. Thus,

$$\sum_{k=0}^{+\infty} E\|x_k\|^\delta \leq \sum_{k=0}^{+\infty} M\gamma^k = \frac{M}{1-\gamma} < +\infty,$$

i.e., the system is stochastically δ-moment stable. Now, assume (4.3) is stochastically δ-moment stable. Let $\xi = \overline{\lim}_{k\to\infty} \|x_k\|$, then from Markov's inequality, we have that for any $c > 0$, the following holds:

$$\begin{aligned} P(\xi \geq c) &= P\left(\cap_{n=1}^{\infty} \cup_{m=n}^{\infty} (\|x_m\| \geq c)\right) \leq P\left(\cup_{m=n}^{\infty}(\|x_m\| \geq c)\right) \\ &\leq \sum_{m=n}^{\infty} P(\|x_m\| \geq c) \leq \frac{1}{c^\delta} \sum_{m=n}^{\infty} E\|x_m\|^\delta \overset{n\to\infty}{\longrightarrow} 0. \end{aligned}$$

Thus with any $c > 0$, $P(\xi \geq c) = 0$, from which we obtain

$$P(\xi > 0) = P\left(\bigcup_{n=1}^{\infty} (\xi > \frac{1}{n})\right) \leq \sum_{n=1}^{\infty} P(\xi > \frac{1}{n}) = 0. \tag{4.4}$$

It follows that $P(\xi = 0) = 1$. This proves that $\lim_{k\to\infty} x_k = 0$ almost surely and the proof of (i) is complete.

For (ii), suppose (4.3) is exponentially $\delta-$moment stable, i.e., there exists $M > 0$ and $0 < \gamma < 1$, such that $E\|x_k\|^\delta \le M\gamma^k$. From Markov's inequality, we obtain for any $\varepsilon > 0$,

$$P(\|x_k\| \ge \varepsilon) \le \frac{1}{\varepsilon^\delta} E\|x_k\|^\delta \le \frac{M}{\varepsilon^\delta}\gamma^k,$$

which implies that (4.3) is exponentially stable in probability.

Assume $\|x_k\| \le \|x_0\| N^k$. If (4.3) is exponentially stable in probability, then for all $\varepsilon > 0$, there exists $0 < \gamma_1 < 1$ and $M(\varepsilon) > 0$ such that

$$P(\|x_k\| \ge \varepsilon) \le M(\varepsilon)\gamma_1^k.$$

It follows that

$$\begin{aligned} E\|x_k\|^\delta &= \int_{(\|x_k\|\ge\varepsilon)} \|x_k\|^\delta P(d\omega) + \int_{(\|x_k\|<\varepsilon)} \|x_k\|^\delta P(d\omega) \\ &\le \|x_0\|^\delta N^{k\delta} P(\|x_k\| \ge \varepsilon) + \varepsilon^\delta \le M(\varepsilon)\|x_0\|^\delta (N^\delta\gamma_1)^k + \varepsilon^\delta. \end{aligned}$$

Since $\lim_{\delta\to 0} N^\delta \gamma_1 = \gamma_1 < 1$, there is $\delta' > 0$ and $\gamma < 1$ such that $N^{\delta'}\gamma_1 < \gamma$. Hence,

$$E\|x_k\|^{\delta'} \le M(\varepsilon)\|x_0\|^{\delta'}\gamma^k + \varepsilon^{\delta'}$$

Therefore, $\lim_{k\to\infty} E\|x_k\|^{\delta'} \le \varepsilon^{\delta'}$. However, as $\varepsilon > 0$ is arbitrary but fixed, this implies that $\lim_{k\to+\infty} E\|x_k\|^{\delta'} = 0$, and the system is δ'-moment stable.

□

Theorem 4.1 established that for the jump linear system (2.1), exponential δ-moment, stochastic δ-moment and δ-moment stability are all equivalent. A direct consequence of the above proposition is the following corollary:

Corollary 4.4: *For the jump linear system (2.1) with a Markov form process, weak exponential stability in probability is equivalent to δ-moment stability for some $\delta > 0$ and they both imply almost sure stability.*

To continue the analysis, we briefly discuss the top Lyapunov exponent for jump linear systems: for the remainder of section 4, the Markovian form process $\{\sigma_k\}$ is assumed to possess a single ergodic (indecomposable) class. We let $\log(\cdot)$ denote the extended real-valued function from $[0, +\infty]$ to $[-\infty, +\infty]$, defined by

$$\log(x) = \begin{cases} \log(x), & \text{if } x \in (0, +\infty), \\ -\infty, & \text{if } x = 0, \\ +\infty, & \text{if } x = +\infty. \end{cases}$$

For system (2.1), we define the *top Lyapunov exponent* for each $\xi \in \Xi$ as the extended real value

$$\alpha_\xi = \overline{\lim_{n\to+\infty}} \frac{1}{n} E_\xi \log \|H(\sigma_n)\dots H(\sigma_0)\| \tag{4.5}$$

and

$$\alpha = \alpha_\pi = \lim_{n\to+\infty} \frac{1}{n} E_\pi \log \|H(\sigma_n)\dots H(\sigma_0)\|, \tag{4.5'}$$

if the indicated limit exists.

Lemma 4.5: *For the system (2.1) with a Markov form process with a single ergodic class,*

(i) For any $\xi \in \Xi$, $\alpha_\xi \in [-\infty, +\infty)$ and $\alpha_\xi = -\infty$ if $P_\xi(A) > 0$ with

$$A \stackrel{\text{def}}{=} \{\omega \in \Omega : \exists \text{ finite } n \text{ such that } \|H(\sigma_n)\dots H(\sigma_0)\| = 0\}.$$

The limit $\alpha = \alpha_\pi$ in (4.5′) exists (possibly infinite).

(ii)

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \log \|H(\sigma_n)\cdots H(\sigma_0)\| = \alpha \qquad P_\pi - a.s.. \tag{4.6}$$

(iii) For any $\xi \in \Xi$ satisfying $\xi << \pi$, we have $\alpha_\xi \le \alpha_\pi$.
Furthermore, if $H(j)$ is nonsingular for all $j \in \underline{N}$ and $\xi << \pi$, then

(iii) $\alpha_\xi = \lim_{n\to+\infty} \frac{1}{n} E_\xi \log \|H(\sigma_n)\dots H(\sigma_0)\|$ is finite and $\alpha_\xi = \alpha_\pi = \alpha$.

(iv) $\lim_{n\to+\infty} \frac{1}{n} \log \|H(\sigma_n)\dots H(\sigma_0)\| = \alpha$, P_π-a.s.. There is a proper subspace L of $\mathcal{R}^n$ such that for any $\xi << \pi$ and $x \in \mathcal{R}^n \backslash L$, we have

$$\overline{\lim_{k\to+\infty}} \frac{1}{k} \log \|H(\sigma_k)\cdots H(\sigma_0)x\| = \alpha, \qquad P_\xi - a.s..$$

Proof: (i). Let $G = \max_{1\le k\le N} \|H(k)\|$ with $H(k) = (h_{ij}(k))$ for all $k \in \underline{N}$. Let $\chi_n = \log \|H(\sigma_n)\cdots H(\sigma_0)\|$. Then,

$$\chi_n \le \log \|H(\sigma_n)\| \cdots \|H(\sigma_0)\| \le \log G^{n+1} = (n+1)\log G. \tag{4.7}$$

It follows that $\alpha_\xi \le \log G < +\infty$. If $P_\xi(A) > 0$, clearly, $\alpha_\xi = -\infty$. Actually, $P_\xi(A) > 0$ implies that there is a $j < +\infty$ such that $P_\xi(A_j) > 0$ with $A_j \stackrel{\text{def}}{=} \{\omega \in \Omega : H(\sigma_k)\cdots H(\sigma_0) = 0, \ \forall k \ge j\}$. It follows that

$$\begin{aligned}\frac{1}{k} E_\xi \chi_k = E_\xi\left(\frac{1}{k}\chi_k\right) &= \int_{A_j} \frac{1}{k}\chi_k P_\xi(d\omega) + \int_{A_j^c} \frac{1}{k}\chi_k P_\xi(d\omega) \\ &\le \frac{k+1}{k}(\log G) P_\xi(A_j^c) + \int_{A_j} \frac{1}{k}\chi_k P_\xi(d\omega) = -\infty, \qquad \forall k > j.\end{aligned}$$

Thus, $\alpha_\xi = \lim_{k\to+\infty} k^{-1}E_\xi\chi_k = -\infty$.

To show the limit α exists, let $a_n = E_\pi \log \|H(\sigma_{n-1})\cdots H(\sigma_0)\|$. We show that a_n is a subadditive sequence, i.e., $a_{m+n} \le a_n + a_m$ for all m and n. Actually, by stationarity,

$$\begin{aligned} a_{n+m} &= E_\pi \log \|H(\sigma_{n+m-1})\dots H(\sigma_0)\| \\ &\le E_\pi \log\{\|H(\sigma_{n+m-1})\dots H(\sigma_n)\|\|H(\sigma_{n-1})\dots H(\sigma_0)\|\} \\ &= E_\pi \log \|H(\sigma_{n+m-1})\dots H(\sigma_n)\| + E_\pi \log \|H(\sigma_{n-1})\dots H(\sigma_0)\| \\ &= a_n + a_m. \end{aligned}$$

If there exists m_0 such that $a_{m_0} = -\infty$, then from (4.7), we have $a_n = -\infty$ for $n \ge m_0$, thus $\alpha = -\infty$. Otherwise, for any n, a_n is finite. For any $m > 0$, and $n = pm + q$, $0 \le q < m$, we have

$$\frac{a_n}{n} = \frac{a_{pm+q}}{pm+q} \le \frac{p}{pm+q}a_m + \frac{a_q}{pm+q}.$$

It follows that $\overline{\lim}_{n\to\infty} \frac{a_n}{n} \le \frac{a_m}{m}$, hence $\overline{\lim}_{n\to\infty} \frac{a_n}{n} \le \underline{\lim}_{m\to\infty} \frac{a_m}{m}$, therefore, $\lim_{n\to\infty} \frac{a_n}{n}$ exists. In fact, we have

$$\begin{aligned} -\infty \le \alpha &= \lim_{n\to\infty} \frac{1}{n} E_\pi \log \|H(\sigma_n)\cdots H(\sigma_0)\| \\ &= \inf_{n\ge 1} \frac{1}{n} E_\pi \log \|H(\sigma_n)\cdots H(\sigma_1)\| \le \log G < +\infty. \end{aligned}$$

This proves (i).

(ii). We need to show (4.6) holds. Suppose that $\{\sigma_k\}$ is stationary, i.e., with initial distribution π. For any $m > 0$ arbitrarily given, let $n = pm+q$, where $0 \le q < m$, we obtain

$$\begin{aligned} \frac{1}{n}\log \|H(\sigma_n)\cdots H(\sigma_0)\| &\le \frac{1}{n}\log \|H(\sigma_n)\cdots H(\sigma_{pm})\| \\ &\quad + \frac{p}{n}\frac{1}{p}\sum_{i=0}^{p-1} \log \|H(\sigma_{(i+1)m-1})\cdots H(\sigma_{im})\|. \end{aligned} \tag{4.8}$$

With the initial distribution π, the process $\{\sigma_k\}$ is a stationary and irreducible Markov chain. Therefore, for any $m > 0$ given, the process $\{\widetilde{\sigma}_i \stackrel{\text{def}}{=} (\sigma_{(i+1)m-1}, \dots, \sigma_{im})\}_{i=0}^{+\infty}$ is also a finite state, time homogeneous, irreducible and stationary Markov chain with a unique invariant distribution $\widetilde{\pi}$. If there exists m_0 such that $E_\pi \log \|H(\sigma_{m_0-1})\cdots H(\sigma_0)\| = -\infty$, then $H(\sigma_{m_0-1})\dots H(\sigma_0) = 0$ with positive (P_π or $P_{\widetilde{\pi}}$) probability and $\alpha = -\infty$. Also, there exists $j_0, j_1, \dots, j_{m_0-1}$ such that $H(j_{m_0-1})\cdots H(j_0) = 0$ and the state $(j_{m_0-1}, \dots, j_0)$ is positive recurrent for the chain $\{\widetilde{\sigma}_i\}$. It follows that for P_π-almost all $\omega \in \Omega$, $\|H(\sigma_n)\cdots H(\sigma_0)\| = 0$ for sufficiently large n, hence $\lim_{n\to\infty} \frac{1}{n}\log\|H(\sigma_n)\cdots H(\sigma_0)\| = -\infty = \alpha$, P_π-a.s. and henceforth (4.6) is valid. Suppose the above does not happen, i.e., for any $m > 0$, $E_\pi \log\|H(\sigma_m)\cdots H(\sigma_0)\|$ is finite. From the Law of Large Numbers, we have

$$\begin{aligned}\lim_{p\to\infty}\frac{1}{p}\sum_{i=0}^{p-1}&\log\|H(\sigma_{(i+1)m-1})\cdots H(\sigma_{im})\|\\ &= E_{\tilde{\pi}}\log\|H(\sigma_{m-1})\cdots H(\sigma_0)\| = E_\pi\log\|H(\sigma_{m-1})\cdots H(\sigma_0)\|.\end{aligned} \tag{4.9}$$

Moreover, $\log\|H(\sigma_n)\cdots H(\sigma_{pm})\| \le (q+1)\log G$ is bounded from above. From (4.8) and (4.9), we obtain

$$\overline{\lim_{n\to\infty}}\frac{1}{n}\log\|H(\sigma_n)\cdots H(\sigma_0)\| \le \frac{1}{m}E_\pi\log\|H(\sigma_{m-1})\cdots H(\sigma_0)\| \quad P_\pi-\text{a.s.}.$$

Hence,

$$\begin{aligned}\overline{\lim_{n\to\infty}}\frac{1}{n}\log\|H(\sigma_n)\cdots H(\sigma_0)\| &\le \inf_{m\ge1}\frac{1}{m}E_\pi\log\|H(\sigma_{m-1})\cdots H(\sigma_0)\|\\ &= \alpha. \qquad P_\pi - \text{a.s.}.\end{aligned} \tag{4.10}$$

Next, let

$$\gamma_n = \frac{1}{n}\sum_{i=0}^{n}\log\|H(\sigma_i)\| - \frac{1}{n}\log\|H(\sigma_n)\cdots H(\sigma_0)\|.$$

Clearly, $\gamma_n \ge 0$. From the Law of Large Numbers, we have

$$\underline{\lim}_{n\to\infty}\gamma_n = E_\pi\log\|H(\sigma_0)\| - \overline{\lim_{n\to\infty}}\frac{1}{n}\log\|H(\sigma_n)\cdots H(\sigma_0)\|$$

and

$$\begin{aligned}E_\pi\gamma_n &= \frac{1}{n}\sum_{i=0}^{n}E_\pi\log\|H(\sigma_i)\| - \frac{1}{n}E_\pi\log\|H(\sigma_n)\cdots H(\sigma_0)\|\\ &\overset{n\to\infty}{\longrightarrow} E_\pi\log\|H(\sigma_0)\| - \alpha.\end{aligned}$$

From Fatou's Lemma, we obtain

$$\begin{aligned}0 \le E_\pi\,\underline{\lim}_{n\to\infty}\gamma_n &= E_\pi\log\|H(\sigma_0)\| - E_\pi\,\overline{\lim_{n\to\infty}}\frac{1}{n}\log\|H(\sigma_n)\cdots H(\sigma_0)\|\\ &\le \underline{\lim}_{n\to\infty}E_\pi\gamma_n = E_\pi\log\|H(\sigma_0)\| - \alpha.\end{aligned}$$

Therefore,

$$E_\pi \overline{\lim_{n\to\infty}} \frac{1}{n} \log \|H(\sigma_n)\cdots H(\sigma_0)\| \geq \alpha. \tag{4.11}$$

From (4.10) and (4.11), we obtain the desired equality

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \log \|H(\sigma_n)\cdots H(\sigma_0)\| = \alpha. \quad P_\pi - \text{a.s.}.$$

This completes the proof of (ii).

(iii). Since $H(j)$ is nonsingular, the condition of Oseledec's Theorem ([29]) is satisfied and the limit in (4.6) exists, i.e.,

$$\lim_{n\to+\infty} \frac{1}{n} \log \|H(\sigma_n)\cdots H(\sigma_0)\| = \alpha = \alpha_\pi, \quad P_\pi - \text{a.s.},$$

which is finite. We need to show that the limit in (4.5) exists and $\alpha_\xi = \alpha$. However, by the nonsingularity of $H(j)$ and (4.7), we obtain

$$-\infty < -2|\log g| \leq \frac{1}{n} \log \|H(\sigma_n)\cdots H(\sigma_0)\| \leq 2|\log G| < +\infty,$$

where $g = \min_{1\leq k\leq N} \|H^{-1}(k)\|^{-1} > 0$. From this, the fact that $P_\xi << P_\pi$ (since $\xi << \pi$), and the Dominated Convergence Theorem, we have

$$\begin{aligned}\alpha_\xi &= \lim_{n\to+\infty} \frac{1}{n} E_\xi \log \|H(\sigma_n)\cdots H(\sigma_0)\| \\ &= E_\xi\{\lim_{n\to+\infty} \frac{1}{n} \log \|H(\sigma_n)\cdots H(\sigma_0)\|\} = \alpha.\end{aligned}$$

(v). The first statement follows from Oseledec's Theorem as in (iv). The second statement follows from the nonrandom spectrum theorem in ([30]). □

Remarks:

(1). In (i), we conjecture that the condition $P_\xi(A) > 0$ is also necessary for $\alpha_\xi = -\infty$. If $\{\sigma_k\}$ is not irreducible, α_ξ may not be a good quantity for almost sure stability. For example, let $H(1) = 0$ and $H(2) = 10$, and $P = \begin{pmatrix} 0.5 & 0.5 \\ 0 & 1 \end{pmatrix}$. It is easy to check that $\alpha_\xi = -\infty < 0$. However, the system (2.1) with the above structure is not almost surely stable. We also conjecture that for any $\xi << \pi$, $\alpha_\xi = \alpha_\pi$ without the invertibility assumption of $H(i)$. Unfortunately, we have not been able to find a rigorous proof for these conjectures.

(2). The top Lyapunov exponent α is closely related to the almost sure stability property of (2.1). Clearly, when $\alpha < 0$, (2.1) is P_π-almost surely stable, and by the fact that there exists a single ergodic class, this implies that the system is almost surely stable (for any initial distribution ξ). When $\alpha > 0$, it is easy to show that (2.1) is not almost surely stable. A question arises at the "bifurcation" point when $\alpha = 0$. We conjecture that when $\alpha = 0$, the system (2.1) is not almost surely stable. A rigorous proof of this is currently under research. However, the following simple example illustrates our intuition that $\alpha = 0$ implies that (2.1) is not almost surely stable.

Example 4.6: Consider a scalar ($n = 1$) jump linear system with an iid form process. Assume $H(i) \neq 0$ for all $i \in \underline{N}$. Let $(p_1, \ldots, p_N)$ be the common distribution of σ_k. In this case, by the Law of Large Numbers, we have

$$\begin{aligned}\alpha &= \lim_{n\to+\infty} \frac{1}{n} \log|H(\sigma_n)\ldots H(\sigma_0)| \\ &= \lim_{n\to+\infty} \frac{1}{n} \sum_{i=0}^{n} \log|H(\sigma_i)| = E\log|H(\sigma_i)| = \sum_{i=1}^{N} p_i \log|H(i)| \quad \text{a.s..}\end{aligned}$$

Suppose that $\alpha = 0$. Define $\eta_k = \log|H(\sigma_k)|$. Then, $\{\eta_k\}$ is an iid process with $E\{\eta_k\} = \alpha = 0$ and $E\{\eta_k^2\} = \sigma^2 < +\infty$. Thus, by the Law of Iterated Logarithm, we obtain

$$P\{\overline{\lim_{n\to+\infty}} \frac{\sum_{k=0}^{n-1} \eta_k}{\sqrt{2\sigma^2 n \log\log n}} = 1\} = 1.$$

It follows that

$$\overline{\lim_{n\to+\infty}} \sum_{k=0}^{n-1} \eta_k = \overline{\lim_{n\to+\infty}} \log|H(\sigma_{n-1})\ldots H(\sigma_0)| = +\infty \qquad \text{a.s..}$$

Therefore, we end up with

$$\overline{\lim_{n\to+\infty}} |x_n(\omega, x_0)| = \overline{\lim_{n\to+\infty}} |H(\sigma_{n-1})\ldots H(\sigma_0)x_0| = +\infty \qquad \text{a.s.}$$

for all $x_0 \neq 0$. The system is not almost surely stable.

Now, we begin to study the relationship between δ-moment and almost sure stability. The δ-moment stability region Σ^δ and the almost sure stability region Σ^a in the parameter space of jump linear system are defined by

$$\Sigma^\delta = \{(H(1), \ldots, H(N)) : (2.1) \text{ is } \delta - \text{moment stable.}\}$$

and

$$\Sigma^a = \{(H(1), \ldots, H(N)) : (2.1) \text{ is almost surely stable.}\},$$

respectively. From the above, we see that we can decompose Σ^a into a disjoint union of the form

$$\Sigma^a = \Sigma^a_- \cup \Sigma^a_0$$

with

$$\Sigma^a_- \stackrel{\text{def}}{=} \Sigma^a \cap \{(H(1), \ldots, H(N)) : \alpha < 0\}$$

and

$$\Sigma^a_0 \stackrel{\text{def}}{=} \Sigma^a \cap \{(H(1), \ldots, H(N)) : \alpha = 0\}$$

If the conjecture given before Definition 4.6 is true, then we expect $\Sigma^a_0 = \phi$, i.e., $\Sigma^a = \Sigma^a_-$. The following theorem is one of our main results:

Theorem 4.7: *For the system (2.1) with a finite state Markov form process, we have*

(i) For any $0 < \delta_1 \leq \delta_2$, $\Sigma^{\delta_2} \subseteq \Sigma^{\delta_1}$. $\Sigma^\delta \subseteq \Sigma^a$ for all $\delta > 0$ and Σ^δ is an open set.

(ii) If $\{\sigma_k\}$ is irreducible, i.e., each pair of states communicates, then

$$\Sigma^a_- = \lim_{\delta \downarrow 0+} \Sigma^\delta \stackrel{\text{def}}{=} \cup_{\delta > 0} \Sigma^\delta \subseteq \Sigma^a.$$

Before we prove this theorem, some comments are in order. The above result simply states that, roughly speaking, the δ-moment stability region Σ^δ is monotonically increasing to the almost sure stability region Σ^a from the interior as δ goes to 0 from above. This is a significant result which generalizes the results reported in Feng et al. ([18]) and Fang et al. ([19]).

To prove the theorem, we establish some fundamental results first.

Lemma 4.8:

(a). For any random variable ξ, the function $F(x)$ defined by

$$F(x) = (E\|\xi\|^x)^{\frac{1}{x}}$$

is nondecreasing on $(0, +\infty)$ whenever it is well defined.

(b). Suppose $\{\sigma_k\}$ is a finite state and irreducible Markov chain with probability transition matrix P, let P_s denote the submatrix of P obtained by deleting the l−th row and the l−th column for some

$l \in \underline{N}$, then the spectral radius $\rho(P_s)$ of P_s is strictly less than one, i.e., there exists a $B > 0$ and $0 \leq r_1 < 1$ such that

$$\|P_s^k\| \leq B r_1^k \quad (\forall k \geq 0).$$

Proof: (a). For any x and y satisfying $0 < x \leq y$, let $\alpha = y/x$, then $\alpha \geq 1$, and $\phi(x) = x^\alpha$ is a convex function of x. Using Jensen's inequality, we have

$$\phi(E\|\xi\|^x) \leq E\phi(\|\xi\|^x)$$

From which, we obtain

$$(E\|\xi\|^x)^{\frac{1}{x}} \leq (E\|\xi\|^y)^{\frac{1}{y}},$$

thus $F(x) \leq F(y)$, which means that $F(x)$ is nondecreasing.

(b). Let $P = (p_{ij})_{N\times N}$. Without loss of generality, we assume that P_s is the submatrix of P obtained by deleting the first row and first column of P, define

$$\bar{P} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & p_{11} & p_{12} & \dots & p_{1N} \\ p_{21} & 0 & p_{22} & \dots & p_{2N} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ p_{N1} & 0 & p_{N2} & \dots & p_{NN} \end{pmatrix} = \begin{pmatrix} X & Y \\ Z & P_s \end{pmatrix},$$

where X,Y and Z are block matrices with appropriate dimensions. It is obvious that $\bar{P}$ is also a stochastic matrix, thus we can form a new $N+1$ state Markov chain $\{\bar{\sigma}_k\}$ with probability transition matrix $\bar{P}$. Since $\{\sigma_k\}$ is an irreducible chain, there exists $i \in \underline{N}\backslash\{1\}$ such that $p_{i1} > 0$, thus $\{\bar{\sigma}_k\}$ is a Markov chain with the absorbing state 1 and the transient states $2, \dots, N+1$, therefore there is no cyclically transferring subclass of states in the ergodic class $\{1\}$ for the Markov chain $\{\bar{\sigma}_k\}$. From [31], we have

$$Q = \lim_{k\to\infty} \frac{1}{k} \sum_{l=1}^{k} \bar{P}^l = \lim_{k\to\infty} \bar{P}^k.$$

Since $2, \dots, N+1$ are transient states of $\{\bar{\sigma}_k\}$, it follows that

$$Q = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \dots & 0 \end{pmatrix}.$$

Moreover, it is easy to prove that

$$\bar{P}^k = \begin{pmatrix} * & * \\ * & * + P_s^k \end{pmatrix},$$

where $*$ denotes a nonnegative matrix because of the nonnegativity of $\bar{P}$. From the structure of Q and the above discussion, $\lim_{k\to\infty} P_s^k = 0$, hence $\rho(P_s) < 1$. The rest of (b) can be easily obtained (see [17]). This completes the proof. □

Lemma 4.9: (Large Deviation Theorem) *Let $\{\sigma_n\}$ be a finite state time homogeneous and irreducible Markov chain with a unique invariant distribution π. For any fixed integer $m > 1$, let*

$$\Phi((j+1)m, jm) = H(\sigma_{(j+1)m-1}) \cdots H(\sigma_{jm}),$$
$$\Delta_m = E_\pi\{\log \|\Phi(m-1, 0)\|.$$

Suppose that for any k, $H(\sigma_k)H(\sigma_{k-1})\cdots H(\sigma_1) \neq 0$, then, for any $\varepsilon > 0$, there exist M,$\delta > 0$ such that

$$P_\pi\left(\frac{1}{p}\sum_{j=0}^{p-1} \log \|\Phi((j+1)m, jm)\| \geq \Delta_m + \varepsilon\right) \leq M\exp(-\delta p).$$

Proof: The proof is given in the appendix. □

Lemma 4.10: *If there exists an $\varepsilon < 0$ such that*

$$\lim_{n\to\infty} P_\xi\left(\frac{1}{n}\log\|H(\sigma_n)\cdots H(\sigma_0)\| \geq \varepsilon\right) = 0,$$

where the convergence is exponential, then there exists a δ such that

$$\lim_{k\to+\infty} E_\xi\|x_k(\omega, x_0)\|^\delta = 0,$$

for any x_0.

Proof: Let $\mathcal{A}, \mathcal{A}^c \in \mathcal{F}$ be defined as

$$\mathcal{A} = \left(\omega \in \Omega \Big| \frac{1}{n}\log\|H(\sigma_n)\cdots H(\sigma_0)\| \geq \varepsilon\right),$$
$$\mathcal{A}^c = \left(\omega \in \Omega \Big| \frac{1}{n}\log\|H(\sigma_n)\cdots H(\sigma_0)\| < \varepsilon\right).$$

With $G = \max_{1\leq j\leq N} \|H(j)\| + 1$, we have

$$\|H(\sigma_n)\cdots H(\sigma_0)\| \leq \|H(\sigma_n)\|\cdots\|H(\sigma_0)\| \leq G^{n+1}.$$

Notice that for any $\omega \in \mathcal{A}^c$,

$$\frac{1}{n}\log\|H(\sigma_n)\cdots H(\sigma_0)\| < \varepsilon \Rightarrow \|H(\sigma_n)\cdots H(\sigma_0)\| \leq e^{\varepsilon n}$$

and by the hypothesis, there are $M_2 > 0$ and $0 < \gamma < 1$ such that

$$P_\xi(\mathcal{A}) \leq M_2\gamma^n.$$

Therefore,

$$\begin{aligned} E_\xi\|H(\sigma_n)\cdots H(\sigma_0)\|^\delta &= \int_{\mathcal{A}} \|H(\sigma_n)\cdots H(\sigma_0)\|^\delta P_\xi(d\omega) + \\ &\quad + \int_{\mathcal{A}^c} \|H(\sigma_n)\cdots H(\sigma_0)\|^\delta P_\xi(d\omega) \\ &\leq G^{\delta(n+1)}P_\xi(\mathcal{A}) + e^{\delta\varepsilon n} \leq M_2 G(G^\delta\gamma)^n + e^{\delta\varepsilon n}. \end{aligned}$$

Since $\lim_{\delta\to 0} G^\delta\gamma = \gamma < 1$, there exists a $\delta > 0$ such that $0 < G^\delta\gamma < 1$. Moreover, as $\varepsilon < 0$, it follows that

$$\lim_{n\to\infty} E_\xi\|H(\sigma_n)\cdots H(\sigma_0)\|^\delta = 0.$$

□

Now, we are ready to prove Theorem 4.7.

Proof of Theorem 4.7:

(i). For any $\delta_2 > \delta_1 > 0$, if $(H(1), H(2), \ldots, H(N)) \in \Sigma^{\delta_2}$, then the system (2.1) is δ_2−moment stable, i.e., $\lim_{k\to\infty} E_\xi\|x_k\|^{\delta_2} = 0$ for all $\xi \in \Xi$. From Lemma 4.8, we have

$$\left(E_\xi\|x_k\|^{\delta_1}\right)^{\frac{1}{\delta_1}} \leq \left(E_\xi\|x_k\|^{\delta_2}\right)^{\frac{1}{\delta_2}}.$$

Then, $\lim_{k\to\infty} E_\xi\|x_k\|^{\delta_1} = 0$ for all $\xi \in \Xi$ and $(H(1), H(2), \ldots, H(N)) \in \Sigma^{\delta_1}$. Hence, $\Sigma^{\delta_2} \subseteq \Sigma^{\delta_1}$.

For any $\delta > 0$, if $(H(1), H(2), \ldots, H(N)) \in \Sigma^\delta$, then (2.1) is δ−moment stable. From Corollary 4.4, the system (2.1) is almost surely stable, i.e., $(H(1), H(2), \ldots, H(N)) \in \Sigma^a$. Thus $\Sigma^\delta \subseteq \Sigma^a$.

Now, we want to prove that Σ^δ is open. For any $(H(1), \ldots, H(N)) \in \Sigma^\delta$, then (2.1) is δ−moment stable, i.e., $\lim_{n\to\infty} E\|H(\sigma_n)\cdots H(\sigma_0)\|^\delta = 0$

for any initial distribution. Using the same notations as in the proof of Theorem 4.1, for $0 < \rho < 1$, there exists an $m > 0$ such that

$$\max_{0\le i\le N} E_{\xi_i} \|H(\sigma_{m-1})\cdots H(\sigma_0)\|^\delta \le \rho.$$

Here, ξ_i is the i-th row of P. For any $r > 0$ with $\rho < r < 1$, because the left side of the above inequality is a continuous function in $(H(1), \dots, H(N))$, there exists an open neighborhood U of $(H(1), \dots, H(N))$ such that for any $(\bar{H}(1), \dots, \bar{H}(N)) \in U$,

$$\max_{0\le i\le N} E_{\xi_i} \|\bar{H}(\sigma_{m-1})\cdots \bar{H}(\sigma_0)\|^\delta \le r < 1.$$

Following the same procedure as in the proof of Theorem 4.1, we can prove that the system (2.1) with the system mode matrices $\bar{H}(1), \dots, \bar{H}(N)$ is exponentially $\delta-$moment stable, hence $(\bar{H}(1), \dots, \bar{H}(N)) \in \Sigma^\delta$, hence $U \in \Sigma^\delta$. This implies that Σ^δ is an open set.

(ii).Recall that the top Lyapunov exponent α satisfies

$$\begin{aligned}&\overline{\lim_{k\to\infty}} \frac{1}{k} \log \|H(\sigma_k)\cdots H(\sigma_0)\| \\ &= \alpha = \lim_{k\to\infty} \frac{1}{k} E_\pi(\log \|H(\sigma_k)\cdots H(\sigma_0)\|) \qquad P_\pi - \text{a.s..}\end{aligned}$$

Thus, if $(H(1), \dots, H(N)) \in \Sigma^a_- = \Sigma^a \cap \{(H(1), \dots, H(N)) : \alpha < 0\}$, then $\alpha < 0$. In this case, it is sufficient to show that there exists a $\delta > 0$ such that (2.1) is $\delta-$moment stable.

Suppose that $\alpha < 0$ (α may be $-\infty$), and with a slight abuse of notation, there exists a finite α and $\varepsilon_0 > 0$ satisfying $\alpha + \varepsilon_0 < 0$ such that

$$\Delta_m \stackrel{\text{def}}{=} E_\pi(\log \|H(\sigma_{m-1})\cdots H(\sigma_0)\|) < m(\alpha + \varepsilon_0). \tag{4.12}$$

Case (a). If there exists an $m > 0$ such that $\Delta_m = -\infty$, with

$$\Delta_m = \sum_{i_1, i_2, \dots, i_m} \pi_{i_1} p_{i_1 i_2} \cdots p_{i_{m-1} i_m} \log \|H(i_m)\cdots H(i_1)\|,$$

there exists $j_1, j_2, \dots, j_m$ such that $P_\pi\{\sigma_{m-1} = j_m, \dots, \sigma_0 = j_1\} > 0$, $\|H(j_m)\cdots H(j_1)\| = 0$, i.e., $H(j_m)\cdots H(j_1) = 0$. In this case, we want to show that there exists a $\delta > 0$ such that (2.1) is $\delta-$moment stable. Let

$G = \max_{1\le k\le N} \|H(k)\|$, we have

$$
\begin{aligned}
&E_\pi \|H(\sigma_{pm-1})\cdots H(\sigma_0)\|^\delta \\
&= \sum_{i_1,i_2,\dots,i_{pm}} \pi_{i_1} p_{i_1 i_2} \cdots p_{i_{pm-1} i_{pm}} \|H(i_{pm})\cdots H(i_1)\|^\delta \\
&\le \sum_{i_1,i_2,\dots,i_{pm}} \pi_{i_1} p_{i_1 i_2} \cdots p_{i_{pm-1} i_{pm}} \|H(i_{pm})\cdots H(i_{(p-1)m+1})\|^\delta \cdots \\
&\|H(i_m)\cdots H(i_1)\|^\delta \\
&\le G^{\delta pm} \sum_{\substack{(i_{rm},\dots,i_{(r-1)m+1})\neq(j_m,\dots,j_1)\\ 1\le r\le p}} \pi_{i_1} p_{i_1 i_2} \cdots p_{i_{pm-1} i_{pm}}.
\end{aligned} \tag{$*$}
$$

We first prove a special case: $m = 1$. Without loss of generality, we can assume that $j_1 = 1$, in this case, the summation on the right hand side of $(*)$ is

$$
\begin{aligned}
&\sum_{\substack{(i_{rm},\dots,i_{(r-1)m+1})\neq(j_m,\dots,j_1)\\ 1\le r\le p}} \pi_{i_1} p_{i_1 i_2} \cdots p_{i_{pm-1} i_{pm}} \\
&= \sum_{i_0,i_1,\dots,i_k=2}^{N} \pi_{i_0} p_{i_0 i_1} \cdots p_{i_{pm-1} i_{pm}} = (\pi_2,\dots,\pi_N) P_s^{pm} c,
\end{aligned}
$$

where $c = (1,\dots,1)'$. From (b) of Lemma 4.8 and $(*)$, there exists $B_1 > 0$ and $0 < r_1 < 1$ such that

$$E_\pi \|H(\sigma_{pm-1})\cdots H(\sigma_0)\|^\delta \le B_1 G^{\delta pm} r_1^{pm} = B_1 (G^\delta r_1)^{pm}.$$

Note, $\lim_{\delta\to 0} G^{\delta m} r_1 = r_1 < 1$ and there exists a $\delta > 0$ such that $G^\delta r_1 < 1$. Thus, for such $\delta > 0$, we have

$$\lim_{p\to\infty} E_\pi \|H(\sigma_{pm})\cdots H(\sigma_0)\|^\delta = 0$$

for $m = 1$. Following a similar procedure as given in the proof of Theorem 4.1, we can conclude that (2.1) is $\delta-$moment stable, thus

$$(H(1),\cdots,H(N)) \in \Sigma^\delta.$$

For $m > 1$, define $x_p = (\sigma_{pm},\dots,\sigma_{(p-1)m+1})$. Because $\{\sigma_k\}$ is an irreducible finite state Markov chain, x_p is also a finite state irreducible Markov chain and $(j_m,\dots,j_1)$ is one state of $\{x_p\}$. Following a similar procedure as given for the $m = 1$ case, there exists $\delta > 0$ such that (2.1) is $\delta-$moment stable, i.e., $(H(1),\dots,H(N)) \in \Sigma^\delta$.

Case (b). If for any m, $\Delta_m > -\infty$, we want to show next that this implies that for some $\varepsilon < 0$,

$$\lim_{n\to\infty} P_\pi \left(\frac{1}{n} \log \|H(\sigma_n)\cdots H(\sigma_0)\| \geq \varepsilon \right) = 0, \tag{4.13}$$

where the convergence is exponential.

For the $m > 0$ satisfying (4.12), let $n = pm + q$, where $p \geq 0$ and $0 \leq q < m$, we then have

$$\begin{aligned}\frac{1}{n}\log\|H(\sigma_n)\cdots H(\sigma_0)\| &\leq \frac{1}{n}\log\|H(\sigma_n)\cdots H(\sigma_{pm})\| \\ &\quad + \frac{1}{n}\log\|H(\sigma_{pm-1})\cdots H(\sigma_0)\| \\ &\leq \frac{1}{n}\sum_{i=pm}^{n} \log\|H(\sigma_i)\| \\ &\quad + \frac{1}{n}\sum_{j=0}^{p-1}\log\|H(\sigma_{(j+1)m-1})\cdots H(\sigma_{jm})\|.\end{aligned}$$

We again use the notation $G = \max_{1\leq j\leq N}\|H(j)\|$, then from the inequality given above we obtain,

$$\begin{aligned}\frac{1}{n}\log\|H(\sigma_n)\cdots H(\sigma_0)\| &\leq \frac{q}{n}\log G \\ &\quad + \frac{p}{n}\frac{1}{p}\sum_{j=0}^{p-1}\log\|H(\sigma_{(j+1)m-1})\cdots H(\sigma_{jm})\|\end{aligned} \tag{4.14}$$

For $\varepsilon < 0$ to be determined later, we have

$$\begin{aligned}&P_\pi\left(\frac{1}{n}\log\|H(\sigma_n)\cdots H(\sigma_0)\| \geq \varepsilon\right) \\ &\leq P_\pi\left(\frac{q}{n}\log G + \frac{p}{n}\frac{1}{p}\sum_{j=0}^{p-1}\log\|H(\sigma_{(j+1)m_1}\cdots H(\sigma_{jm})\| \geq \varepsilon\right) \\ &= P_\pi\left(\frac{1}{p}\sum_{j=0}^{p-1}\log\|H(\sigma_{(j+1)m_1}\cdots H(\sigma_{jm})\| \geq \frac{n}{p}(\varepsilon - \frac{q}{n}\log G)\right).\end{aligned} \tag{4.15}$$

Notice that for fixed m, $\lim_{p\to\infty}\frac{n}{p} = m$, and $\lim_{p\to\infty}\frac{n}{p}(\varepsilon - \frac{q}{n}\log G) = m\varepsilon$.

Define $\widetilde{\sigma}_j = (\sigma_{(j+1)m-1}, \ldots, \sigma_{jm})$ for $j = 0, 1, 2, \ldots$. Since $\{\sigma_k\}$ is a finite state, time homogeneous and irreducible Markov chain, $\{\widetilde{\sigma}_j\}_{j=0}^{+\infty}$ is also a finite state, time homogeneous and irreducible Markov chain. Let $\widetilde{\pi}$ denote the unique invariant distribution of $\{\widetilde{\sigma}_j\}$. Let $\underline{N}^m \overset{\text{def}}{=} \underline{N} \times \ldots \times \underline{N}$ and $f : \underline{N}^m \longrightarrow \mathcal{R}$ be defined as

$$f(\hat{\sigma}) = \log \|H(\hat{\sigma}_{m-1}) \cdots H(\hat{\sigma}_0)\|, \quad \hat{\sigma} = (\hat{\sigma}_{m-1}, \ldots, \hat{\sigma}_0) \in \underline{N}.$$

Then, we have

$$\begin{aligned} E_{\widetilde{\pi}} f(\widetilde{\sigma}_0) &= E_{\widetilde{\pi}} \log \|H(\sigma_{m-1}) \cdots H(\sigma_0)\| \\ &= E_{\pi} \log \|H(\sigma_{m-1}) \cdots H(\sigma_0)\| = \Delta_m. \end{aligned} \tag{4.16}$$

Since $\alpha + \varepsilon_0 < 0$, there exists an $\varepsilon_1 > 0$ such that $\alpha + \varepsilon_0 + 2\varepsilon_1 < 0$. Choose $\varepsilon = \alpha + \varepsilon_0 + 2\varepsilon_1$. Then, there exists a $K > 0$ such that for any $n \geq K$, we have $\frac{n}{p}(\varepsilon - \frac{q}{n} \log G) \geq m(\varepsilon - \varepsilon_1)$.

Then, from (4.12), (4.15) and (4.16), we obtain for $n \geq K$,

$$\begin{aligned} & P_{\pi}\left(\frac{1}{n} \log \|H(\sigma_n) \cdots H(\sigma_0)\| \geq \varepsilon\right) \\ & \overset{(4.15)}{\leq} P_{\pi}\left(\frac{1}{p} \sum_{j=0}^{p-1} f(\widetilde{\sigma}_j) \geq m(\varepsilon - \varepsilon_1)\right) \\ & = P_{\widetilde{\pi}}\left(\frac{1}{p} \sum_{j=0}^{p-1} f(\widetilde{\sigma}_j) \geq m(\alpha + \varepsilon_0) + m\varepsilon_1\right) \\ & \overset{(4.13),(4.16)}{\leq} P_{\widetilde{\pi}}\left(\frac{1}{p} \sum_{j=0}^{p-1} f(\widetilde{\sigma}_j) \geq \Delta_m + m\varepsilon_1\right). \end{aligned} \tag{4.17}$$

Due to the fact that $\Delta_m > -\infty$ for any m, it is easy to verify that $H(\sigma_k)H(\sigma_{k-1}) \cdots H(\sigma_k) \neq 0$ for any k. From the large deviation theorem (Lemma 4.9), there exists $M_1 > 0$ and $0 < \gamma_1 < 1$, such that for p large,

$$P_{\widetilde{\pi}}\left(\frac{1}{p} \sum_{j=0}^{p-1} f(\widetilde{\sigma}_j) \geq \Delta_m + m\varepsilon_1\right) \leq M_1 \gamma_1^p. \tag{4.18}$$

Thus, for n large, from (4.17) and (4.18), we obtain

$$\begin{aligned} P_{\pi}\left(\frac{1}{n} \log \|H(\sigma_n) \cdots H(\sigma_0)\| \geq \varepsilon\right) & \leq M_1 \gamma_1^p \\ & = (M_1 \gamma_1^{-q/m})(\gamma_1^{1/m})^{pm+q} \leq M_2 \gamma^n \end{aligned}$$

where $\gamma = \gamma_1^{1/m}$ and $M_2 = M_1\gamma_1^{-1}$. This establishes (4.13) for the case when $\alpha \neq -\infty$.

Now, from Lemma 4.10 and (4.13), there exists a $\delta > 0$ such that

$$\lim_{k\to+\infty} E_\pi \|x_k(\omega, x_0)\|^\delta = 0, \qquad \forall x_0 \in \mathcal{R}^n. \tag{4.19}$$

Since $\{\sigma_k\}$ is irreducible and $\pi > 0$, from Lemma 2.3, the system (2.1) is $\delta-$moment stable for any initial distribution $\xi \in \Xi$.

Summarizing the above, we have proved that if $(H(1), \ldots, H(N)) \in \Sigma_-^a$, then there exists a $\delta_0 > 0$ such that $(H(1), \ldots, H(N)) \in \Sigma^{\delta_0}$. Notice that Σ^δ (as a set) is monotonically increasing as δ decreases. Therefore we conclude that

$$\Sigma_-^a \subseteq \lim_{\delta\to 0} \Sigma^\delta = \cup_{\delta>0}\Sigma^\delta \subseteq \Sigma^a.$$

Next, we will show that the first set containment $\subseteq$ can be replaced by a set equality $=$. In fact, if $(H(1), \ldots, H(N)) \in \Sigma^\delta$ for $\delta > 0$, then (2.1) is $\delta-$moment stable. From Theorem 4.1, we know that (2.1) is also exponentially $\delta-$moment stable, i.e., there exists an $M > 0$ and $0 < \rho < 1$ such that $E\|H(\sigma_n)\cdots H(\sigma_1)\|^\delta \le M\rho^n$, thus we have

$$g(\delta) \stackrel{\text{def}}{=} \lim_{n\to\infty} \frac{1}{n} \log E\|H(\sigma_n)\cdots H(\sigma_0)\|^\delta \le \log\rho < 0.$$

From Jensen's inequality, we obtain that

$$\begin{aligned}\delta\alpha &= \lim_{n\to\infty} \frac{1}{n} E \log \|H(\sigma_n)\cdots H(\sigma_0)\|^\delta \\ &\le \lim_{n\to\infty} \frac{1}{n} \log E\|H(\sigma_n)\cdots H(\sigma_0)\|^\delta \\ &\le \log\rho < 0.\end{aligned}$$

Thus, $\alpha < 0$, i.e., $(H(1), \ldots, H(N)) \in \Sigma_-^a$. This completes the proof of (ii). □

Next, we will investigate the relationship between the top Lyapunov exponent and the $\delta-$moment Lyapunov exponent. Similar results to those obtained by Arnold et al. ([21],[28]) for stochastic differential equations of the Ito type are obtained.

Recall that the top Lyapunov exponent as defined in (4.5) is given by

$$\alpha_\xi = \overline{\lim_{n\to\infty}} \frac{1}{n} E_\xi \log \|H(\sigma_n)\cdots H(\sigma_0)\|,$$

which has the properties given in Lemma 4.5. The *$\delta-$moment Lyapunov exponent* can be defined as the extended real-valued function $\beta(\cdot,\cdot) : \mathcal{R} \times \Xi \longrightarrow \overline{\mathcal{R}} = [-\infty, +\infty]$ given by

$$\beta(\delta, \xi) = \lim_{n\to+\infty} \frac{1}{n} \log E_\xi \|H(\sigma_n)\cdots H(\sigma_0)\|^\delta, \tag{4.20}$$

if the indicated limit exists. We may also define

$$\overline{\beta}(\delta,\xi) = \overline{\lim_{n\to+\infty}} \frac{1}{n} \log E_\xi \|H(\sigma_n)\cdots H(\sigma_0)\|^\delta$$
$$\underline{\beta}(\delta,\xi) = \underline{\lim}_{n\to+\infty} \frac{1}{n} \log E_\xi \|H(\sigma_n)\cdots H(\sigma_0)\|^\delta. \tag{4.20'}$$

Define $\beta(0,\xi) = \underline{\beta}(0,\xi) = \overline{\beta}(0,\xi) = 0$. For simplicity, we assume that the matrices $H(j)$ are nonsingular for all $j \in \underline{N}$. We have the following properties of the δ-moment Lyapunov exponent(s):

Proposition 4.11: *Assume that the $H(j)$ are invertible matrices for all $j \in \underline{N}$. For any $\xi \in \Xi$, we have*

(i) $-\infty < \underline{\beta}(\delta,\xi) \le \overline{\beta}(\delta,\xi) < +\infty$.

(ii) $\delta\alpha_\xi \le \overline{\beta}(\delta,\xi)$. *If $\xi << \pi$, then,* $\delta\alpha_\xi = \delta\alpha_\pi \le \underline{\beta}(\delta,\xi)$.

Proof: (i). Since $H(1),\ldots,H(N)$ are nonsingular, let

$$G = \max_{1\le k\le N} \|H(k)\|,\quad g = \min_{1\le k\le N} \|H^{-1}(k)\|^{-1},\ \text{with } \|H^{-1}(k)\|^{-1} \le \|H(k)\|$$

for any k, we have for $\delta > 0$,

$$-\infty < -2\delta|\log g| \le \frac{1}{n}\log E_\xi\|H(\sigma_n)\cdots H(\sigma_0)\|^\delta \le 2\delta|\log G| < +\infty,$$

and for $\delta < 0$,

$$-\infty < 2\delta|\log G| \le \frac{1}{n}\log E_\xi\|H(\sigma_n)\cdots H(\sigma_0)\|^\delta \le -2\delta|\log g| < +\infty.$$

These inequalities imply that $-\infty < \underline{\beta}(\delta,\xi) \le \overline{\beta}(\delta,\xi) < +\infty$.

(ii). $\log x$ is a concave function on $(0,+\infty)$, and using Jensen's inequality, we obtain

$$\frac{1}{n}E_\xi \log\|H(\sigma_n)\cdots H(\sigma_0)\|^\delta \le \frac{1}{n}\log E_\xi\|H(\sigma_n)\cdots H(\sigma_0)\|^\delta.$$

Taking the limit superum on both sides, we have $\delta\alpha_\xi \le \overline{\beta}(\delta,\xi)$. If $\xi << \pi$, taking the limit infimum and applying Lemma 4.5, we have $\delta\alpha_\xi = \delta\alpha_\pi \le \underline{\beta}(\delta,\xi)$. □

Lemma 4.12:

(i) Suppose that $f(x)$ is continuous, and for any x_1, x_2,

$$f\left(\frac{x_1+x_2}{2}\right) \le \frac{f(x_1)+f(x_2)}{2}$$

holds, then $f(x)$ is convex.

(ii) If $f(x)$ is convex and the domain of f is an open set, then f is also continuous.

Proof: This is a well-known result from calculus. □

Theorem 4.13: *If $H(1), \ldots, H(N)$ are invertible, then*

(i) For any $\xi \in \Xi$ fixed, $\overline{\beta}(\cdot, \xi)$ is a convex function defined on $\mathcal{R}$.

(ii). For any $\xi \in \Xi$ fixed, $\overline{\beta}(\delta, \xi)/\delta$ is nondecreasing on $\mathcal{R}\backslash\{0\}$. Let $\dot{\overline{\beta}}(\delta, \xi) = \frac{d}{d\delta}\underline{\beta}(\delta, \xi)$, then $\dot{\overline{\beta}}(0^-, \xi) \le \alpha_\xi \le \dot{\overline{\beta}}(0^+, \xi)$ and $\dot{\overline{\beta}}(0, \xi) = \alpha_\xi$ whenever it exists.

(iii) For any $\delta \in \mathcal{R}$ fixed, $\underline{\beta}(\delta, \cdot)$ is a concave function defined on Ξ.

Proof:

(i) Define $f(x) = \log E_\xi \|H(\sigma_n) \cdots H(\sigma_0)\|^x$. Then for any $x_1, x_2 \in \mathcal{R}$, from the Cauchy-Schwartz inequality, we have

$$
\begin{aligned}
&f(\frac{x_1 + x_2}{2}) \\
&= \log \left\{ E_\xi \left[(\|H(\sigma_n) \cdots H(\sigma_0)\|^{x_1})^{1/2} (\|H(\sigma_n) \cdots H(\sigma_0)\|^{x_2})^{1/2} \right] \right\} \\
&\le \log \left\{ (E_\xi \|H(\sigma_n) \cdots H(\sigma_0)\|^{x_1})^{1/2} (E_\xi \|H(\sigma_n) \cdots H(\sigma_0)\|^{x_2})^{1/2} \right\} \\
&= \frac{f(x_1) + f(x_2)}{2}.
\end{aligned}
\tag{4.21}
$$

It is easy to show that $f(x)$ is continuous on $\mathcal{R}$ from the invertibility of $H(1), \ldots, H(N)$. From (4.21) and Lemma 4.12, we know that $f(x)$ is convex. Thus for any $0 \le \lambda \le 1$, we have

$$f(\lambda x_1 + (1 - \lambda)x_2) \le \lambda f(x_1) + (1 - \lambda) f(x_2).$$

Multiplying both sides of the inequality by $1/n$ and taking the limit supremum, we obtain

$$\overline{\beta}(\lambda x_1 + (1 - \lambda)x_2, \xi) \le \lambda \overline{\beta}(x_1, \xi) + (1 - \lambda)\overline{\beta}(x_2, \xi),$$

which implies that $\overline{\beta}(x, \xi)$ is convex.

(ii) Since $\log x$ is monotonically increasing on $(0, +\infty)$, from Lemma 4.8, we have that $\overline{\beta}(\delta, \xi)/\delta$ is nondecreasing on $(0, +\infty)$. When δ takes values in $(-\infty, 0)$, replacing δ by $-\delta$ and using Lemma 4.8, we obtain that $\overline{\beta}(\delta, \xi)/\delta$ is also nondecreasing on $(-\infty, 0)$. By (ii) of Proposition 4.11, we have for any δ, $\delta\alpha_\xi \le \overline{\beta}(\delta, \xi)$. Thus, for $\delta > 0$, we have

$$\frac{\overline{\beta}(-\delta, \xi)}{-\delta} \le \alpha_\xi \le \frac{\overline{\beta}(\delta, \xi)}{\delta}.$$

Taking the limit $\delta \downarrow 0^+$, we have (ii).

(iii) For $\lambda \in [0,1]$ and $\xi, \zeta \in \Xi$, it is easy to verify that $P_{\lambda\xi+(1-\lambda)\zeta} = \lambda P_\xi + (1-\lambda)P_\zeta$. Then, since $\log(\cdot)$ is concave, we obtain

$$\begin{aligned} &\frac{1}{n}\log E_{\lambda\xi+(1-\lambda)\zeta}\|H(\sigma_n)\cdots H(\sigma_0)\|^\delta \\ &= \frac{1}{n}\log(\lambda E_\xi\|H(\sigma_n)\cdots H(\sigma_0)\|^\delta \\ &\quad + (1-\lambda)E_\zeta\|H(\sigma_n)\cdots H(\sigma_0)\|^\delta) \\ &\geq \frac{\lambda}{n}\log E_\xi\|H(\sigma_n)\cdots H(\sigma_0)\|^\delta \\ &\quad + \frac{1-\lambda}{n}\log E_\zeta\|H(\sigma_n)\cdots H(\sigma_0)\|^\delta. \end{aligned}$$

Taking the limit infimum on both sides, we have

$$\underline{\beta}(\delta, \lambda\xi + (1-\lambda)\zeta) \geq \lambda\underline{\beta}(\delta,\xi) + (1-\lambda)\underline{\beta}(\delta,\zeta),$$

i.e., $\underline{\beta}(\delta,\cdot)$ is a concave function. □

Remark: Theorem 4.13 is also valid for a more general class of systems as long as the indicated expectations exist. This is evident from our proof. In the theorem we require that the $\delta-$moment top Lyapunov exponent is differentiable at zero. In fact, we conjecture that it is analytic for the system (2.1). Details on this aspect of our work will be reported at a later date. For the one dimensional case, let $\|H(i)\| = a_i$ $(i = 1, 2, \ldots, N)$, then from [19], we obtain

$$\beta(\delta) = \log(p_1 a_1^\delta + p_2 a_2^\delta + \cdots + p_N a_N^\delta).$$

If at least one of the a_i is not equal to zero, this is an analytic function of δ at $\delta = 0$. Thus, the conjecture is true for one dimensional jump linear systems of type (2.1) with an iid form process $\{\sigma_k\}$.

Additional properties of $\delta-$moment Lyapunov exponents are still under research. Parallel stability results for continuous-time jump linear systems of the type (1.2) will be presented in a separate paper.

5 Examples

Example 5.1: (Almost sure stability does not imply second moment stability)

Let $H(1) = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$, $H(2) = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$ for systems of the type (2.1) with iid jumps, $p_1 = p_2 = 0.5$.

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$H(1)$ and $H(2)$ are nilpotent matrices with $H(1)^2 = H(2)^2 = 0$, and it is easy to show that (2.1) is almost surely stable. From the results of Ji et al. [11], it is easy to check that (2.1) is not second moment stable.

Example 5.2: (Almost sure stability does not imply that the individual modes are stable)

Let $H(1) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$, $H(2) = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$ for systems of type (2.1) with an iid form process σ_k with $p_1 = p_2 = 0.5$.

For any $m, n \geq 1$, $H(1)^m H(2)^n = 0$, and the system is almost surely stable. But $H(1)$ and $H(2)$ are not stable.

Example 5.3: (A general illustrative example)

Let $H(1) = \widetilde{\alpha}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $H(2) = \widetilde{\alpha}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ for systems of type (2.1) with an iid form process σ_k with $p_1 = p_2 = 0.5$.

In this example, we study the stability properties of the system (2.1) as $\widetilde{\alpha}$ varies in the interval $(0, 1]$.

(i). $\|H(1)\|_2 = \|H(2)\|_2 = \left(\frac{\sqrt{5}+1}{2}\right)\widetilde{\alpha}$, using the result from [19] we obtain that if $\widetilde{\alpha} < \frac{\sqrt{5}-1}{2}$, the system (2.1) is almost surely stable.

(ii). Next we use Theorem 3.5 to study the almost sure stability: Let $x = (\cos\theta, \sin\theta)^T$, and let $f(\theta) = \|H(1)x\|_2^2 \|H(2)x\|_2^2$. Then

$$\begin{aligned} f(\theta) &= \widetilde{\alpha}^4 \left\| \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \right\|_2^2 \left\| \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \right\|_2^2 \\ &= \widetilde{\alpha}^4 \left((\cos\theta + \sin\theta)^2 + \sin^2\theta\right)\left(\cos^2\theta + (\cos\theta + \sin\theta)^2\right), \\ &= \widetilde{\alpha}^4 (2 + 3\sin 2\theta + \frac{5}{4}\sin^2 2\theta) \end{aligned}$$

and $\max f(\theta) = \frac{25}{4}\widetilde{\alpha}^4$, so $\sigma_{max} = \sqrt{\frac{5}{2}}\widetilde{\alpha}$. From Theorem 3.5, (2.1) is almost surely stable if $\widetilde{\alpha} < \sqrt{\frac{2}{5}}$. This is an improved estimate of the almost sure stability region.

(iii). From the mean value stability criterion ([19]), we know that (2.1) is mean value stable if and only if $p_1 H(1) + p_2 H(2)$ is a stable matrix. From this we obtain that if $\widetilde{\alpha} < 2/3$, (2.1) is mean value stable. By the remark given after Lemma 3.2, we obtain that (2.1) is almost surely stable if $\widetilde{\alpha} < 2/3$.

(iv). Although the proof in (iii) is simple, the following approach seems to be applicable to more general cases.

Define

$$G_k = \begin{pmatrix} G_k^1 & G_k^2 \\ G_k^3 & G_k^4 \end{pmatrix} = H(\sigma_k)\cdots H(\sigma_0), H(\sigma_k) = \widetilde{\alpha}\begin{pmatrix} 1 & \delta_k \\ 1-\delta_k & 1 \end{pmatrix},$$

where $\delta_k = 1$ if $\sigma_k = 1$ and $\delta_k = 0$ if $\sigma_k = 2$.

Let Σ_k is the sum of all the entries of G_k. Then it is easy to show that

$$\Sigma_k = \widetilde{\alpha}\left(\Sigma_{k-1} + \delta_k(G_{k-1}^3 + G_{k-1}^4) + (1-\delta_k)(G_{k-1}^1 + G_{k-1}^2)\right). \tag{5.1}$$

Since $\{\sigma_k\}$ is an iid sequence, δ_k and $1-\delta_k$ are independent of G_{k-1}^j $(1 \leq j \leq 4)$ and $E\delta_k = E(1-\delta_k) = 0.5$. Thus from (5.1), let $m_k = E\Sigma_k$, we have

$$\begin{aligned} m_k &= \widetilde{\alpha}\left(E\Sigma_{k-1} + E\delta_k E(G_{k-1}^3 + G_{k-1}^4)E(1-\delta_k)E(G_{k-1}^1 + G_{k-1}^2)\right) \\ &= \widetilde{\alpha}\left(m_{k-1} + \frac{1}{2}(E(G_{k-1}^3 + G_{k-1}^4) + E(G_{k-1}^1 + G_{k-1}^2))\right) \\ &= \widetilde{\alpha}(m_{k-1} + \frac{1}{2}E\Sigma_{k-1}) = \frac{3}{2}\widetilde{\alpha}m_{k-1} = \left(\frac{3}{2}\widetilde{\alpha}\right)^k m_0. \end{aligned}$$

Let $\mathcal{F}_k$ be the smallest $\sigma-$algebra generated by $\sigma_k, \ldots, \sigma_0$. Then it is easy to show that

$$E\left(\Sigma_{k+1}|\mathcal{F}_k\right) = \left(\frac{3}{2}\widetilde{\alpha}\right)\Sigma_k.$$

Thus, if $\widetilde{\alpha} < 2/3$, then $E(\Sigma_{k+1}|\mathcal{F}_k) \leq \Sigma_k$, which implies that $\{\Sigma_k, \mathcal{F}_k\}$ is a supermartingale. From the Martingale Convergence Theorem ([27]), there exists a random variable Σ such that $\lim_{k\to\infty}\Sigma_k = \Sigma$, almost surely.

Next, we want to prove that $\Sigma = 0$. In fact, for $c > 0$, we have

$$\begin{aligned} P(\Sigma > 0) \leq P\left(\cup_{k=m}^{\infty}(\Sigma_k > c)\right) &\leq \sum_{k=m}^{\infty} P(\sigma_k > c) \\ &\leq \sum_{k=m}^{\infty}\frac{1}{c}E\Sigma_k \leq \frac{1}{c}\sum_{k=m}^{\infty}\left(\frac{3}{2}\widetilde{\alpha}\right)^k m_0. \end{aligned} \tag{5.2}$$

If $\widetilde{\alpha} < 2/3$, then $\sum_{k=1}^{\infty}(\frac{3}{2}\widetilde{\alpha})^k$ is a convergent series. In (5.2), let m go to infinity, we obtain $P(\Sigma > c) = 0$. Thus

$$P(\Sigma > 0) \leq \sum_{k=1}^{\infty} P(\Sigma > \frac{1}{k}) = 0,$$

so $P(\Sigma > 0) = 0$. Since $\Sigma \geq 0$, we therefore have

$$P(\Sigma = 0) = 1.$$

From which we obtain that if $\widetilde{\alpha} < 2/3$, then (2.1) is almost surely stable.

(v). In [19], we have proved that if $EH(\sigma_1)^T H(\sigma_1)$ is stable, then (2.1) is second moment stable. Since

$$EH(\sigma_1)^T H(\sigma_1) = \frac{1}{2}(H(1)^T H(1) + H(2)^T H(2)) = \frac{\widetilde{\alpha}^2}{2}\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$$

and its eigenvalues are $5\widetilde{\alpha}^2/2$ and $\widetilde{\alpha}^2/2$, if $\widetilde{\alpha}^2/2 < 1$, i.e. $\widetilde{\alpha} < \sqrt{0.4}$, (2.1) is second moment stable, so it is almost surely stable (same as (ii)).

(vi). The Kronecker product is a good tool for studying the second moment stability. It is easy to show that (2.1) with an iid form process is second moment stable if and only if $p_1 H(1)\otimes H(1) + \cdots + p_N H(N)\otimes H(N)$ is stable. A variation of this is the following:

Let $P_k = Ex_k x_k^T$, (2.1) is second moment stable if and only if P_k is a matrix sequence which converges to zero. For the present example we have been studying, we have

$$\begin{aligned} P_{k+1} &= \frac{1}{2}\left(H(1)P_k H(1)^T + H(2)P_k H(2)^T\right) \\ &= \frac{\widetilde{\alpha}^2}{2}\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} P_k \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} P_k \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right). \end{aligned}$$

Let $P_k = \begin{pmatrix} {p_k}^1 & {p_k}^2 \\ {p_k}^2 & {p_k}^3 \end{pmatrix}$ and $y_k = ({p_k}^1, {p_k}^2, {p_k}^3)^T$, then

$$y_{k+1} = \frac{\widetilde{\alpha}^2}{2}\begin{pmatrix} 2 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 2 \end{pmatrix} y_k \triangleq R y_k.$$

(2.1) is second moment stable if and only if R is a stable matrix, i.e., all eigenvalues of R have modulus less than unity. The eigenvalues of R are $-\widetilde{\alpha}^2/2$ and $\frac{5\pm\sqrt{17}}{4}\widetilde{\alpha}^2$. Thus if $\frac{5+\sqrt{17}}{4}\widetilde{\alpha}^2 < 1$, i.e., $\widetilde{\alpha} < \sqrt{\frac{5-\sqrt{17}}{2}}$, (2.1) is second moment stable.

Remark: Notice that $\sqrt{\frac{5-\sqrt{17}}{2}} < \frac{2}{3}$, and we know that $\widetilde{\alpha} < \frac{2}{3}$ is sufficient for almost sure stability. Hence, criteria for almost sure stability and second moment stability can differ greatly, and almost sure stability does not necessarily imply second moment stability.

(vii). Let $M(G)$ be the largest entry of a matrix G. Notice that $H(1)^m = \widetilde{\alpha}^m \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$ and $H(2)^m = \widetilde{\alpha}^m \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}$, then

$$M(H(1)^m H(2)^n) \geq \widetilde{\alpha}^{m+n} mn$$

and

$$M(H(2)^m H(2)^n) \geq \widetilde{\alpha}^{m+n} mn.$$

Let

$$H(\sigma_n)\cdots H(\sigma_1) = \widetilde{\alpha}^n \begin{pmatrix} 1 & l_m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ l_{m-1} & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & l_1 \\ 0 & 1 \end{pmatrix},$$

or the variations with l_m or l_1 at the opposite positions in the corresponding matrices, where $l_i > 0$, $\sum_{i=1}^m l_i = n$, and $m = m(n)$ is a random sequence and $m \to \infty$ if $n \to \infty$. Since $\{\sigma_n\}$ is an iid sequence with the probability distribution $(0.5, 0.5)$, $\{l_i\}$ is also an iid sequence with the distribution $P\{l_i = k\} = 0.5^k$ $(k > 0)$. It is also easy to show that

$$M(H(\sigma_n)\cdots H(\sigma_1)) \geq \widetilde{\alpha}^n l_1 l_2 \cdots l_m.$$

Thus

$$\frac{1}{m}\log M(H(\sigma_n)\cdots H(\sigma_1)) \geq \frac{1}{m}\sum_{i=1}^m \log l_i + \left(\frac{1}{m}\sum_{i=1}^m l_i\right)\log\widetilde{\alpha}.$$

From the Law of Large Numbers, we have

$$\lim_{m\to\infty}\sum_{i=1}^m \log l_i = E\log l_1 = \sum_{k=1}^\infty \frac{\log k}{2^k},$$
$$\lim_{m\to\infty}\frac{1}{m}\sum_{i=1}^m l_i = El_1 = \sum_{k=1}^\infty \frac{k}{2^k}.$$

Then, noting that $m = m(n)$, we obtain

$$\lim_{n\to\infty}\frac{1}{m}\log M(H(\sigma_n)\cdots H(\sigma_1)) \geq \sum_{k=1}^\infty \frac{\log k}{2^k} + \left(\sum_{k=1}^\infty \frac{k}{2^k}\right)\log\widetilde{\alpha}.$$

So if it is positive, i.e.,

$$\widetilde{\alpha} > \exp\left(-\frac{\sum_{k=1}^\infty \frac{\log k}{2^k}}{\sum_{k=1}^\infty \frac{k}{2^k}}\right) = 0.7758,$$

then (2.1) is almost surely unstable.

This example shows that although the individual modes are stable, i.e., $H(1)$ and $H(2)$ are stable matrices (with $\widetilde{\alpha} < 1$), for $\widetilde{\alpha} > 0.7758$, the jump linear system (2.1) is not almost surely stable, let alone second moment stable. It is well known that it is very easy to construct an example of a finite state Markov chain jump linear system whose individual modes are stable, but the system is not almost surely stable. However, it is difficult to give a similar example for a jump linear system with an iid form process, the above example accomplishes this objective.

(viii). Let $\Pi(A)$ denote the product of all the entries of matrix A and let $G_k = H(\sigma_k)\cdots H(\sigma_0)$, where the notation for entries of G_k is the same as before, let $\Pi_k = \Pi(G_k)$. As before, we can obtain

$$\Pi_{k+1} = \widetilde{\alpha}^4(G_k^1 + G_k^3)(G_k^2 + G_k^4)((1-\delta_k)G_k^1 + \delta_k G_k^3)((1-\delta_k)G_k^2 + \delta_k G_k^4).$$

Let $\mathcal{F}_k$ be the smallest $\sigma-$algebra generated by $\{\sigma_{k-1},\ldots,\sigma_0\}$,then from the inequality $a+b \geq 2\sqrt{ab}$ for $a,b\geq 0$, we obtain

$$\begin{aligned} E(\Pi_{k+1}|\mathcal{F}_k) &= \widetilde{\alpha}^4(G_k^1 + G_k^3)(G_k^2 + G_k^4)\left[\frac{1}{2}(G_k^1G_k^2 + G_k^3G_k^4)\right] \\ &\geq \widetilde{\alpha}^4(2\sqrt{G_k^1G_k^3})(2\sqrt{G_k^2G_k^4})\sqrt{G_k^1G_k^2G_k^3G_k^4} \\ &= 4\widetilde{\alpha}^4 G_k^1G_k^2G_k^3G_k^4 = 4\alpha^4\Pi_k. \end{aligned}$$

From a similar argument, we also have

$$E\Pi_{k+1}^\delta \geq (4\widetilde{\alpha}^4)^\delta (E\Pi_k^\delta)^{\frac{1}{\delta}}.$$

Thus, if $4\widetilde{\alpha}^4 > 1$, i.e., $\widetilde{\alpha} > \sqrt{0.5} = 0.7071$, $\lim_{k\to\infty} E\Pi_k^\delta = +\infty$ for any $\delta > 0$.

If we use the 2−norm, then $\|A\| = \sqrt{\lambda_{max}(A^TA)}$. Thus for any $x \in R^n$, we have $x^TA^TAx \leq \|A\|^2x^Tx$. Choose $x = e_i = (0,\ldots,0,1,0,\ldots,0)^T$, we have $\sum_{k=1}^n a_{ik}^2 \leq \|A\|^2$, from which we obtain that $\max_{i,j}|a_{ij}| \leq \|A\|$. For nonnegative matrix A, $\Pi(A) \leq \|A\|^{n^2}$. Thus, for our problem, we have $\Pi_k \leq \|H(\sigma_k)\cdots H(\sigma_1)\|^4$. Therefore, we obtain

$$\left(E\|H(\sigma_k)\cdots H(\sigma_1)\|^{4\delta}\right)^{1/\delta} \geq \left(E\Pi_k^\delta\right)^{1/\delta} \geq \cdots \geq C(4\widetilde{\alpha}^4)^k,$$

from which we have

$$\frac{1}{k}\log E\|H(\sigma_k)\cdots H(\sigma_1)\|^{4\delta} \geq \frac{1}{k}\delta\log C + \delta\log(4\widetilde{\alpha}^4).$$

Taking the limit supremum, we obtain

$$\overline{\beta}(4\delta,\pi) \geq \delta\log(4\widetilde{\alpha}^4),$$

where $\overline{\beta}(\delta,\pi)$ is the top $\delta-$moment Lyapunov exponent. From §4, we know that $\overline{\beta}(0,\pi) = \gamma$, where γ is the top Lyapunov exponent (if the derivative exists). Thus, if $4\widetilde{\alpha}^4 > 1$, i.e., $\widetilde{\alpha} > 0.7071$, $\gamma > 0$, which means that (2.1) is exponentially unstable almost surely.

This result is better than (vii). The techniques developed in §4 are potentially important for the study of almost sure stability. This will be a topic of our future research.

Summarizing the above for this example, we have

(a). (2.1) is second moment stable if and only if $0 \leq \widetilde{\alpha} < \sqrt{\frac{5-\sqrt{17}}{2}}$;
(b). (1.1) is mean value stable if and only if $0 \leq \widetilde{\alpha} < \frac{2}{3}$;
(c). (1.1) is almost surely stable if $0 \leq \widetilde{\alpha} < \frac{2}{3}$;
(d). (1.1) is almost surely unstable if $\widetilde{\alpha} > 0.7071$.

For $\frac{2}{3} \leq \widetilde{\alpha} \leq \frac{\sqrt{2}}{2}$, we have not found a rigorous method to determine the stability property of the system, this is currently under investigation.

6 Conclusions

We have studied the stability properties of discrete time linear systems which are subject to abrupt changes in structure or linear feedback control systems which have communication delays in the feedback path. Some criteria of almost sure stability of these systems are obtained. After introducing the concept of $\delta-$moment stability, we have shown that all $\delta-$moment stability properties are equivalent, which is a generalization of the results for second moment stability. It is then proved that the region of $\delta-$moment stability is monotonically converging to the region of almost sure stability as $\delta \downarrow 0^+$. This provides a new approach to the study of almost sure stability as the examples show. We have also presented some new results concerning the relationship between the top Lyapunov exponent and top $\delta-$moment Lyapunov exponent.

We have used a large deviation principle in our study of almost sure stability. An interesting future research direction would be a study of the large deviation properties of a random matrix product. The analyticity of the top $\delta-$moment top Lyapunov exponent may also be of theoretical interest.

Appendix

Proof of Theorem 4.9:

The large deviation result of Lemma 4.9 is a consequence of Thereom IV.1 of [32] and we will verify the hypothesis of [32] with the aid of a similar procedure as in [23]. Let $\Phi((j+1)m, jm)$ be defined as in Lemma 4.9. Define

$$Y_p = \sum_{j=0}^{p-1} \log \|\Phi((j+1)m, jm)\|,$$

$$c_p(\delta) = \frac{1}{p} \log E_\pi\{\exp(\delta Y_p)\}.$$

Then we have

$$c_p(\delta) = \log E_\pi \prod_{j=0}^{p-1} \|\Phi((j+1)m, jm)\|^\delta.$$

We first show that $c(\delta) = \lim_{p\to+\infty} c_p(\delta)$ exists for all $\delta \in \mathcal{R}$ (where the limit is possibly be $+\infty$).

Let S denote the state space of the Markov chain $\{\sigma_k\}$. For any sequence $l = (i_0, i_1, \ldots, i_{m-1}) \in S^m$ and $\delta \in \mathcal{R}$, write

$$\Lambda(l;\delta) = \Lambda(i_0, i_1, \ldots, i_{m-1}; \delta) = \|H(i_{m-1})H(i_{m-2})\cdots H(i_0)\|^\delta.$$

Then, we have $\Lambda(l;\delta) > 0$ due to the fact that $H(\sigma_k)\cdots H(\sigma_0) \neq 0$ for any $k \geq 0$. Since $\{\sigma_k\}$ is irreducible, it follows that the chain $\widetilde{r}_j = (\sigma_{(j+1)m-1}, \ldots, \sigma_{jm})$ for $j = 0, 1, \ldots$ is also an irreducible Markov chain with state space

$$\widetilde{S} = \{(i_0, \ldots, i_{m-1}) \in S^m : P_\pi(\sigma_{k+m-1} = i_{m-1}, \ldots, \sigma_k = i_0) > 0, \quad \text{for some } k \geq 0\}.$$

Suppose that $|\widetilde{S}| = \widetilde{N}$ and that we have ordered the states in a certain way so that for $k \in \{1, 2, \ldots, \widetilde{N}\}$, $\Lambda(k;\delta)$ is defined accordingly. Now, consider that

$$\begin{aligned}
E_\pi\{\prod_{j=0}^{p-1} \|\Phi((j+1)m, jm)\|^\delta\} &= \sum_{i_0,\ldots,i_{pm-1}} \pi_{i_0} p_{i_0 i_1} \cdots p_{i_{pm-2} i_{pm-1}} \\
&\times \prod_{j=0}^{p-1} \|\Phi((j+1)m, jm)\|^\delta \\
&= \sum_{i_0,\ldots,i_{pm-1}} \pi_{i_0} p_{i_0 i_1} \cdots p_{i_{pm-2} i_{pm-1}} \Lambda(i_0, i_1, \ldots, i_{m-1}; \delta) \\
&\times \Lambda(i_m, i_{m+1}, \ldots, i_{2m-1}; \delta) \ldots \Lambda(i_{(p-1)m}, i_{(p-1)m+1}, \ldots, i_{pm-1}; \delta) \\
&= E_{\hat{\pi}}\{\prod_{j=0}^{p-1} \Lambda(\widetilde{r}_j; \delta)\} = \sum_{l_0,\ldots,l_{p-1}} \hat{\pi}_{l_0} \hat{p}_{l_0 l_1} \cdots \hat{p}_{l_{p-2} l_{p-1}} \prod_{l=0}^{p-1} \Lambda(l;\delta) \\
&= \sum_{j_0,\ldots,j_{p-1}} (\hat{\pi}_{j_0} \Lambda(j_0;\delta))(\hat{p}_{j_0 j_1} \Lambda(j_1, \delta)) \ldots (\hat{p}_{j_{p-2} j_{p-1}} \Lambda(j_{p-1};\delta)) \\
&= x^T (B(\delta))^{p-2} y.
\end{aligned} \tag{A.1}$$

Where in (A.1) $\hat{P} = (\hat{p}_{jl})_{\widetilde{N}\times\widetilde{N}}$ is the transition matrix of $\widetilde{r}_k$, $\hat{\pi} = (\hat{\pi}_1, \ldots, \hat{\pi}_{\widetilde{N}})$ is its initial distribution induced by π, and

$$\begin{aligned} 0 \neq x^T &\stackrel{\text{def}}{=} (\hat{\pi}_1\Lambda(1;\delta), \ldots, \hat{\pi}_{\widetilde{N}}\Lambda(\widetilde{N};\delta)) \geq 0 \\ y^T &= (1, 1, \ldots, 1) > 0 \\ B(\delta) &= (\hat{p}_{jl}\Lambda(l;\delta))_{\widetilde{N}\times\widetilde{N}} \geq 0. \end{aligned}$$

Since the transition matrix $\hat{P}$ is irreducible and $\Lambda(l;\delta) > 0$, we see that $B(\delta)$ is an irreducible nonnegative matrix. Next, we show that with $c_p(\delta) = \log x^T B(\delta)^{p-2} y$, $c(\delta) = \lim_{p\to\infty} \frac{1}{p} c_p(\delta)$ exists and is differentiable at any point where it is defined. Since $B(\delta)$ is an irreducible nonnegative matrix, from matrix theory [17], there exists a positive vector v such that $B(\delta)v = \rho(B(\delta))v$. Since v and y are positive vectors, there exists positive numbers $L > 0$ and $U > 0$ such that $Lv \leq y \leq Uv$ and $x^T v > 0$, where the last inequality follows from the nonnegativity of $x \neq 0$. Thus, we have

$$\begin{aligned} c_p(\delta) \leq \log x^T B^{p-2}(Uv) &= \log\{U x^T \rho(B(\delta))^{p-2} v\} \\ &= \log U + (p-2)\log\rho(B(\delta)) + \log x^T v, \end{aligned}$$

from which we obtain that

$$\overline{\lim_{p\to\infty}} \frac{1}{p} c_p(\delta) \leq \log\rho(B(\delta)). \tag{A.2}$$

In a similar way, we have

$$c_p(\delta) \geq \log L + (p-2)\log\rho(B(\delta)) + \log x^T v,$$

from which we have

$$\underline{\lim}_{p\to\infty} \frac{1}{p} c_p(\delta) \geq \log\rho(B(\delta)). \tag{A.3}$$

(A.2) and (A.3) yield

$$c(\delta) \stackrel{\text{def}}{=} \lim_{p\to\infty} \frac{1}{p} c_p(\delta) = \log\rho(B(\delta)).$$

Due to the continuity of $B(\delta)$, $\mathcal{D}(c) \stackrel{\text{def}}{=} \{\delta \in \mathcal{R} : c(\delta) < +\infty\}$ is a nonempty open interval containing 0, and c is a closed convex function. Furthermore, because $B(\delta)$ is irreducible, $\rho(B(\delta))$ is simple [17]. Due to the differentiability of $B(\delta)$, it follows that $c(\delta)$ is differentiable [33]. Therefore, From Theorem IV.1 of [32], we obtain that there exists τ such that for any $\varepsilon > 0$ there is $\eta(\varepsilon) > 0$ such that

$$P_\pi \left(|\frac{1}{p} Y_p - \tau| \geq \varepsilon \right) \leq \exp(-\eta p) \qquad (A.4)$$

for p large. However, by the Law of Large Numbers, we should have $\tau = \Delta_m$. Thus, Lemma 4.9 follows from (A.4) directly. □

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