Pergamon

PII: S0016–0032(96)00108–1

J. Franklin Inst. Vol. 335B, No. 1, pp. 163–177, 1998

0016–0032/98 \$19.00+0.00

Global Properties for a Class of Dynamical Neural Circuits

by YUGUANG FANG *and* THOMAS G. KINCAID

Department of Electrical and Computer Engineering, Boston University, 44 Cummington Street, Boston, MA 02215, U.S.A.

(Received 5 August 1996; accepted 27 September 1996)

ABSTRACT: *In this paper, we study the global properties of a class of asymmetrical Hopfield-type neural circuits. We first present a result for the existence and uniqueness of an equilibrium point; this result does not assume smoothness of the neural activation functions. Then we give some testable sufficient conditions for the global stability of such neural circuits. These results generalize a few previous known results and remove some restrictions on the neural circuits.* © 1997 Published by Elsevier Science Ltd

I. Introduction

Hopfield-type neural circuits have been intensively studied in the past decade and have been applied to optimization problems and specific problems of A/D converter design (**1**–**4** and references therein). However, a problem normally encountered in this approach is the existence of more than one equilibrium point, which may correspond to local minima. Global properties are hard to extract and a global minimum is difficult to achieve. To overcome this dilemma, it is desirable to design neural circuits which have only one unique equilibrium point and are globally stable (or attractive) so that the global property can be extracted. In this case, we do not need to specify the initial conditions of the neural circuits, since all trajectories starting from anywhere will settle down to the same unique equilibrium. This equilibrium depends only on the external stimuli. In fact, if the interconnection among individual neurons, the time constants of the circuits and the activation functions are fixed, we obtain a mapping from the external stimuli space to the activation space. Moreover, unlike the Winner-Take-All circuits where resetting of activations has to be made whenever input stimuli change, if the neural circuits are globally stable for each external stimulating input, we need not reset the activations when changing inputs. This is convenient for a neural circuit running in real time.

The qualitative properties of dynamical neural networks (notably Hopfield-type neural networks) have been intensively investigated by Michel and his colleagues (**3**–**6**). They applied large-scale system techniques to obtain a large set of sufficient conditions for local asymptotic (exponential) stability for a few classes of dynamical neural networks. Hirsch (**7**) pointed out the importance of global stability or global attractiveness, and obtained a few sufficient conditions using Gershgorin's circle theorem. Kelly (**8**) applied the contraction mapping technique to obtain some sufficient conditions for global stability. Matsuoka (**9**) generalized some of Hirsch's and Kelly's results using a new Lyapunov function. Recently, Kaszkurewicz and Bhaya (**10**) proved that the diagonal stability of the interconnection matrix implies the existence and uniqueness of an equilibrium and global stability of the equilibrium. Forti *et al.* (**11**) showed that the negative semidefiniteness of the interconnection matrix guarantees the global stability of the Hopfield network with a certain robustness property.

In this paper, we consider a class of asymmetrical Hopfield-type networks whose activation functions are not necessarily continuously differentiable, whose interconnection matrix is not necessarily symmetric or stable. We obtain a more general sufficient ('almost' necessary) condition for the existence and uniqueness of a global equilibrium point, then show that diagonal semistability of a certain matrix implies the global stability of the neural circuits. Some sufficient conditions for global stability are discussed.

II. Notation and Preliminaries

Before we present our main results, we first give some notation and preliminary results which will be used in the sequel. For any matrices A and B, let $A \leqslant_e B (A <_e B)$ denote the elementwise inequalities. For any symmetric matrices A and B, let $A \leqslant B (A < B)$ denote that $B - A$ is positive semidefinite (definite) matrix. $\lambda_i(A)$ denotes the eigenvalue of a matrix A. A^t denotes the matrix transpose for any matrix A. $\mathrm{tr}(A)$ and $\det(A)$ denote the trace and determinant of A, respectively. A matrix A is said to be stable if the real parts of its eigenvalues are negative. $\mathrm{diag}\{d_1, \ldots, d_n\}$ denotes the diagonal matrix with diagonal entries $d_1, \ldots, d_n$. If all diagonal entries of A are positive numbers, we call A a positive diagonal matrix. Let R^n be the n-dimensional real Euclidean space. For a, $b \in R^1$, let $I(a, b)$ denote the interval between a and b. For a function (mapping) F, let F^{-1} denote the inverse function (mapping).

Definition 2.1: For each positive integer n and each pair of n-vectors α and β whose components α_i and β_i lie in the extended real number system, with $\alpha <_e \beta$ (i.e. with $-\infty \leqslant \alpha_i < \beta_i \leqslant \infty$ for $i = 1, 2, \ldots, n$), let $\mathscr{F}^n(\alpha, \beta; R^n)$ denote the set of functions from $I(\alpha_1, \beta_1) \times \cdots \times I(\alpha_n, \beta_n)$ onto R^n defined by: $F \in \mathscr{F}^n(\alpha, \beta; R^n)$ if and only if there exist continuous strictly monotonically increasing functions $f_i(x_i)$ from $I(\alpha_i, \beta_i)$ to R^1 such that for each $x = (x_1, \ldots, x_n)^t \in I(\alpha_1, \beta_1) \times \cdots \times I(\alpha_n, \beta_n)$, $F(x) = (f_1(x_1), \ldots, f_n(x_n))^t$. Define similarly the set of functions from R^n onto $I(\alpha_1, \beta_1) \times \cdots \times I(\alpha_n, \beta_n)$ as $\mathscr{F}(R^n; \alpha, \beta)$: $F \in \mathscr{F}(R^n; \alpha, \beta)$ if and only if $F^{-1} \in \mathscr{F}(\alpha, \beta; R^n)$. In fact, $\mathscr{F}^n(R^n; \alpha, \beta)$ can be defined to be the following set:

$$\left\{ \begin{pmatrix} f_1(x_1) \\ \vdots \\ f_n(x_n) \end{pmatrix} \middle| \, f_i(x_i) \text{ is continuous, strictly increasing, } \beta_i = \max_{v \in R^1} f_i(v), \alpha_i = \min_{v \in R^1} f_i(v) \right\}.$$

Definition 2.2: (**12**) $\mathscr{P}_0$ (respectively, $\mathscr{P}$) denotes the class of square matrices A all of whose principal minors are nonnegative (positive).

Lemma 2.1: [Sandberg and Wilson (**12**)] For n-vectors $\alpha <_e \beta$ whose components lie in the extended real number system, if A is $n \times n$ real matrix, then the equation $F(x)+Ax = B$ has a unique solution for each $F \in \mathscr{F}^n(\alpha, \beta; R^n)$ and each $B \in R^n$ if and only if $A \in \mathscr{P}_0$. □

Definition 2.3: A square matrix A is diagonally semistable (respectively, stable) if there exists a positive diagonal matrix P such that the matrix $PA + A^tP$ is positive semidefinite (definite), i.e. there exists a positive semidefinite (definite) matrix Q such that the Lyapunov equation $PA + A^tP = -Q$ has a positive diagonal matrix solution P.

III. Existence and Uniqueness of the Equilibrium

The circuit model we study in this paper is described as (**10**)

$$C_i \dot{u}_i = -\frac{u_i}{R_i} + \sum_{j=1}^{n} T_{ij} g_j(u_j) + I_i; \quad i = 1, 2, \ldots, n \tag{1}$$

where $\dot{u}_i$ denotes the derivative (we always use overdot to denote the differentiation in this paper), $C_i > 0$ are the neuron amplifier input capacitances, $R_i > 0$ are the resistances, $T = (T_{ij})$ is the $n \times n$ network interconnection matrix, I_i are the (constant) external stimulating currents, $u_i = u_i(t)$ are the neural voltages representing the activity of the i-th neuron, $g_i(u_i)$ are the neural activation functions satisfying: $G = (g_1(u_1), \ldots, g_n(u_n))^t \in \mathscr{F}(R^n; \alpha, \beta)$ for certain n-vectors α and β. Let $B = (I_1, \ldots, I_n)^t$, $G(x) = (g_1(x_1), \ldots, g_n(x_n))^t$, $C = \text{diag}\{C_1, \ldots, C_n\}$, $R = \text{diag}\{R_1, \ldots, R_n\}$ and $u = u(t) = (u_1(t), \ldots, u_n(t))^t$. Obviously, when T is a symmetric matrix, the network model is the well-known Hopfield circuit. Here, we are not restricted to this case.

For a dynamical system to be globally asymptotically stable, a necessary condition is an existence and uniqueness of the equilibrium point. It is useful, therefore, to give some sufficient conditions for the model to have a unique equilibrium point. When the activation functions $g_i(\cdot)$ are continuously differentiable and their derivatives are positive, Forti *et al.* (**11**) obtained a result regarding this issue. Their proof can not be easily applied to the case when the activation functions are not continuously differentiable. However, applying the results of Sandberg and Wilson (**12**), we generalize their result to this case. We have:

Theorem 3.1: Given any n-vectors $\alpha <_e \beta$ whose components lie in the extended real number system, the neural circuit (1) has a unique equilibrium point for any $G(u) = (g_1(u_1), \ldots, g_n(u_n))^t \in \mathscr{F}^n(R^n; \alpha, \beta)$ and any $B \in R^n$ if and only if $-T \in \mathscr{P}_0$.

Proof: Redefine the state variables: $x_i = R_i^{-1} u_i$ and $h_i(x_i) = g_i(u_i) = g_i(R_i x_i)$, $H(x) = (h_1(x_1), \ldots, h_n(x_n))^t$. Obviously, $H(x) \in \mathscr{F}^n(R^n; \alpha, \beta)$ if and only if $G(u) \in \mathscr{F}^n(R^n; \alpha, \beta)$, and the neural network (1) has a unique equilibrium point if and only if the following algebraic equation has a unique solution:

$$-x + TH(x) + B = 0. \tag{2}$$

This equation can be rewritten by the following variable substitution $y = H(x)$ as the following equivalent form:

$$H^{-1}(y)+(-T)y = B. \tag{3}$$

It is easy to verify that there exists a unique solution of Eq. (2) if and only if there exists a unique solution of Eq. (3) [applying the strictly monotonically increasing property of the activation functions]. Also notice that $G(x)\in\mathscr{F}^n(R^n;\alpha,\beta)$ if and only if $H^{-1}(y)\in\mathscr{F}^n(\alpha,\beta;R^n)$. From Lemma 2.1, there exists a unique solution of Eq. (3) for any $H^{-1}(y)\in\mathscr{F}^n(\alpha,\beta;R^n)$ and any $B\in R^n$ if and only if $-T\in\mathscr{P}_0$. From this, the proof of Theorem 3.1 is easily completed. □

Remarks:

(1) In **(11)**, it is proved that a mapping is a diffeomorphism under the smoothness condition. However, when the activation function is not differentiable or if $\dot{g}_i(x_i)$ is not positive, the proof there is not valid. One simple example is $g_i(x_i) = x_i^3$, this is a continuously differentiable strictly monotonically increasing function. However, $\dot{g}_i(0) = 0$, not positive. We can not use Forti *et al.*'s result **(11)**. However, Theorem 3.1 is still applicable.

(2) The proof for the existence and uniqueness of an equilibrium point in **(10)** used the condition that T is invertible. Theorem 3.1 does not imply this assumption. The activation functions are much more general than in **(10)**. In particular, the activation functions in Theorem 3.1 are not required to be bounded, therefore, they can be linear functions.

(3) As we mentioned in the Introduction, the condition for existence and uniqueness of an equilibrium point is "almost" necessary as we witnessed in Theorem 3.1. It is not necessary in the sense that when the activation functions and external inputs are given, the network (1) may have a unique equilibrium point although $-T$ not $\mathscr{P}_0$ matrix. For example, for one neuron case (i.e. $n=1$), let $R=1$, $B=-2$, $G(x)=1/(1+e^{-x})$ and $T=1$. It is obvious that the circuit (1) has a unique equilibrium -1.866, however, $-T$ is not $\mathscr{P}_0$ matrix. The reason for this is because the necessary and sufficient condition is independent of any activation functions and any external inputs, which can be regarded as a kind of robustness property: If $-T\in\mathscr{P}_0$, then there exists a unique equilibrium point no matter what kind of continuous strictly increasing activation functions and external inputs are chosen, this may be very helpful in practical designs.

If more general sufficient conditions for the uniqueness of the equilibrium are desired, it must be related to the specific property of the activation functions, as well as the interconnection matrix. In this case, we obtain the following result.

Theorem 3.2: Suppose that the activation $g_i(x_i)$ are bounded continuous monotonically nondecreasing functions satisfying the following condition:

$$0 \leqslant \gamma_i \leqslant \frac{g_i(x_i)-g_i(y_i)}{x_i-y_i} \leqslant \omega_i < \infty, \quad i = 1,2,\ldots,n$$

for certain nonnegative numbers γ_i and ω_i and any $x_i\in R^1$ and $y_i\in R^1$ satisfying $x_i \neq y_i$. Let $\Gamma = \mathrm{diag}\{\gamma_1,\ldots,\gamma_n\}$ and $\Omega = \mathrm{diag}\{\omega_1,\ldots,\omega_n\}$. Then the neural network (1) has a unique equilibrium point if $\det(I-RTD)\neq 0$ for any nonnegative diagonal matrix D satisfying $\Gamma \leqslant_e D \leqslant_e \Omega$. If the activation functions are assumed to be strictly monotonically increasing, the sufficient condition can be substituted by: det $(I-RTD) \neq 0$ for any positive diagonal matrix D satisfying $\Gamma \leqslant {}_{e}D \leqslant {}_{e}\Omega$.

Proof: Let $\|\cdot\|$ denote the Euclidean norm. Since the activation functions are all bounded, there exists positive number $M>0$ such that $\|G(x)\| \leqslant M$. Let $\Delta = \|RT\|M + \|RB\|$, $\mathrm{O} = \{x \| \|x\| \leqslant \Delta\}$ and $\phi(x) = RTG(x) + RB$. Since $\|\phi(x)\| \leqslant \|RT\| \, \|G(x)\| + \|RB\| \leqslant \Delta$, $\phi(\cdot)$ is a continuous mapping from O into O, and O is a convex bounded closed set in R^n. Then, from Schauder's Fixed Point Theorem **(13)**, there exists a fixed point for $\phi(x) = x$, hence an equilibrium point of **(1)**.

Next, we need to prove that the equilibrium point of (1) is unique. If there exists x and y in R^n such that they are both the equilibrium points of the system (1), i.e. $-R^{-1}x + TG(x) + B = 0$ and $-R^{-1} + TG(x) + B = 0$, subtracting the second from the first, we obtain

$$-R^{-1}(x-y) + T(G(x) - G(y)) = 0. \tag{4}$$

Define a new diagonal matrix $D = \mathrm{diag}\{d_1, \ldots, d_n\}$:

$$d_i = \begin{cases} \dfrac{g_i(x_i) - g_i(y_i)}{x_i - y_i}, & \text{if } x_i \neq y_i \\ \tau_i & \text{otherwise} \end{cases}$$

where τ_i is any number satisfying $\gamma_i \leqslant \tau_i \leqslant \omega_i$. Then, obviously, $\Gamma \leqslant {}_{e}D \leqslant {}_{e}\Omega$. From Eq. (4), we can easily obtain $(x-y) - RT(G(x) - G(y)) = [I - RTD](x-y) = 0$, thus, if $\det(I-RT) \neq 0$, which is guaranteed by the assumption, then $x = y$. This proves the uniqueness of the equilibrium point.

If the activation functions are all strictly monotonically increasing, from the choice of D above, we know that D is a positive diagonal matrix, thus the second part of the theorem is straightforward. □

Remarks:

(1) This theorem does not assume that the activation functions are strictly monotonically increasing, therefore Theorem 3.2 can be applied to some interesting cases like hardlimiting neural circuits. If $-T \in \mathscr{P}_0$, then for any positive diagonal matrix Δ, $\det(\Delta - T) \neq 0$, therefore, $\det(I-RTD) = \det(R)\det((RD)^{-1} - T)$ $\det(D) \neq 0$. This shows that Theorem 3.2 is a better tool for determining the uniqueness of an equilibrium point.

(2) The boundedness and continuity of the activation functions is not needed for the uniqueness proof. Thus, for the network (1) with monotonically nondecreasing (respectively, strictly increasing) activation functions, if $\det(I-RTD) \neq 0$ for any nonnegative (positive) diagonal matrix D satisfying $\Gamma \leqslant {}_{e}D \leqslant {}_{e}\Omega$, then the network (1) has at most one equilibrium point.

(3) The sufficient conditions developed in Theorem 3.2 are also independent of the external input B. It seems that global properties such as the global stability and uniqueness of an equilibrium point are generically independent of such additive external (constant) stimulating input, and that it only affects the location of the equilibrium point.

We use the following simple example to illustrate our results.

Example 3.1. Consider the one-neuron model:

$$\dot{x} = -x + Tg(x) + b \tag{5}$$

where $R = 1$, T and b are constant scalers and $g(x)$ is the activation function.

If $g(x)$ is a sigmoidal function given by $g(x) = 1/(1+\exp(-\lambda x))$, this is a smooth strictly monotonically increasing function. We have $0 \leqslant (g(x)-g(y))/(x-y) \leqslant \max_{v\in R^1}\dot{f}(v) = \lambda/4$. From Theorem 3.2, if $T < 4/\lambda$, the system (5) has a unique equilibrium point. It can be graphically observed that this condition is the necessary and sufficient condition for (5) to have a unique equilibrium point. Notice that when $T = 1$ and $\lambda = 1$, Theorem 3.2 is applicable. However, $-T$ is not $\mathscr{P}_0$ matrix, i.e. Theorem 3.1 can not be applied.

If $g(x) = \lambda x^+ = \lambda \max(0, x)$, where $\lambda > 0$. This function is unbounded and not differentiable. It is obvious that $0 \leqslant (g(x)-g(y))/(x-y) \leqslant \lambda$. From Theorem 3.2, we obtain if $T < 1/\lambda$, the system (5) has at most one equilibrium point. It can be graphically verified that this is the necessary and sufficient condition for (5) to have a unique equilibrium point. From this example, we can also see that the boundedness for existence of an equilibrium point can not be removed in general, for example, when we choose $T = \lambda = B = 1$, then there is no equilibrium of (5) [of course, here, the sufficient condition in Theorem 3.2 is violated].

If $g(x) = \lambda/2(|x+1| - |x-1|)$ where $\lambda > 0$, this is a bounded continuous non-decreasing function and not differentiable. This activation function is often used in the cellular neural circuits (**14**). It is easy to show that $0 \leqslant (g(x)-g(y))/(x-y) \leqslant \lambda$. From Theorem 3.2, the system (5) has a unique equilibrium point if $T < 1/\lambda$. In fact, this condition is also necessary.

IV. Global Stability of the Equilibrium

In the last section, we presented the conditions for the existence and uniqueness of the equilibrium point. A natural question to ask is whether the equilibrium point is asymptotically (globally) stable or not. In (**11**), Forti *et al.* obtained a global stability result for symmetric neural network. For asymmetrical neural network (1), Kaszkurewicz and Bhaya (**10**) gave another sufficient condition: the interconnection matrix T is diagonally stable. A necessary condition for diagonal stability is the stability of T, therefore, their results can not be applied to the popular case where there are no self-interactions in the network (for example the Hopfield network), this is because that the trace of T in this case is zero, which implies that T is not even stable. We generalize their result to include this important case. We have

Theorem 4.1: If the interconnection matrix T is diagonally semistable, then for any $R_i > 0$, any C_i, any external inputs I_i and any continuous strictly monotonically increasing activation functions $g_i(u_i)$ on R^1, the neural circuit (1) is globally asymptotically stable.

Proof: The proof follows the same approach as in (**10**) with some modification. Let $\sigma_i = \inf_{v\in R^1} g_i(v)$ and $\beta_i = \sup_{v\in R^1} g_i(v)$, the infimum and supremum of $g_i(u_i)$, respectively, $i = 1, 2, \ldots, n$, then $G(u) = (g_1(u_1), \ldots, g_n(u_n))^t \in \mathscr{F}^n(R^n; \alpha, \beta)$.

First, we want to prove that $-T \in \mathscr{P}_0$. If T is diagonally semistable, then there exists a positive diagonal matrix P and a positive semidefinite matrix Q such that

$$PT + T^t P = -Q. \tag{6}$$

For any positive number τ, we have (we reserve I to denote the identity matrix with appropriate dimension):

$$P(T-\tau I) + (T-\tau I)P = -Q - 2\tau P = -(Q+2\tau P),$$

and $Q+2\tau P$ is positive definite, hence $T-\tau I$ is diagonally stable. From **(15)**, $-(T-\tau I) = -T+\tau I \in \mathscr{P}$. let τ go to zero, then it is easy to verify that $-T \in \mathscr{P}_0$.

From Theorem 3.1, the neural network (1) has a unique equilibrium point, say u^*, thus the system (1) is equivalent to the following system:

$$\dot{x}_i = \frac{1}{C_i}\left[\frac{1}{R_i}x_i - \sum_{j=1}^{n} T_{ij}\phi_j(x_j)\right], \quad (i = 1, 2, \ldots, n) \tag{7}$$

where $x_i = u_i - u_i^*$, $\phi_i(x_i) = g_i(x_i + u_i^*) - g_i(u_i^*)$ and the new activation function $\phi_i(\cdot)$ is continuous, strictly monotonically increasing. Moreover, $\phi_i(x_i)x_i = [g_i(x_i+u_i^*) - g_i(u_i^*)] \times [(x_i+u_i^*) - u_i^*] > 0$ for $x_i \neq 0$. Let $\Phi(x) = (\phi_1(x_1), \ldots, \phi_n(x_n))^t$, then Eq. (7) can be written as the following matrix form:

$$\dot{x} = -(RC)^{-1} + C^{-1}T\Phi(x). \tag{8}$$

It suffices to prove that Eq. (8) is globally asymptotically stable. Let the positive diagonal matrix $\Pi = \mathrm{diag}\{\pi_1, \ldots, \pi_n\}$, where $\pi_i > 0$ is to be determined. Let

$$V(x) = 2\sum_{i=1}^{n} \pi_i \int_0^x \phi_i(\tau)\,d\tau.$$

Obviously, since $\phi_i(\tau)$ is strictly increasing and $\phi_i(0) = 0$, $\int_0^{x_i} \phi_i(\tau)\,d\tau > 0$ for $x_i \neq 0$, hence $V(x) > 0$ for $x \neq 0$. If $|x_i| \geqslant M > 0$, then we have

$$\int_0^{x_i} \phi_i(\tau)\,d\tau > \begin{cases} \displaystyle\int_0^{M} \phi_i(\tau)\,d\tau + \phi_i(M)(x_i - M), & x_i > 0 \\ \displaystyle\int_0^{-M} \phi_i(\tau)\,d\tau + \phi_i(-M)(x_i + M). & x_i < 0. \end{cases}$$

We conclude that $\lim_{\|x\|\to\infty} V(x) = \infty$, i.e. $V(x)$ is radially unbounded. Moreover,

$$\dot{V}(x) = \Phi^t(x)[\Pi(C^{-1}T) + (C^{-1}T)^t\Pi]\Phi(x) - 2\Phi^t(x)\Pi(CR)^{-1}x. \tag{9}$$

From Eq. (6), $PC(C^{-1}T) + (C^{-1}T)^t(CP) = -Q$. Let $\Pi = PC$, noticing that C and P are both diagonal, we have $\Pi(C^{-1}T) + (C^{-1}T)^t\Pi = -Q$ and Π is a positive diagonal matrix. Let $\Pi(CR)^{-1} = \mathrm{diag}\{r_1, \ldots, r_n\}$, where $r_i > 0$ (in fact, $r_i = p_i/R_i$), from Eq. (9) we obtain

$$\dot{V}(x) = -\Phi^t(x)Q\Phi(x) - 2\Phi^t(x)\Pi(CR)^{-1}x$$

$$\leqslant -2\sum_{i=1}^{n} r_i\phi_i(x_i)x_i < 0 \quad \text{(for any } x \neq 0\text{)},$$

because $\phi_i(x_i)x_i > 0$ for any $x_i \neq 0$. From Lyapunov stability theorem (**16**), the system (8), hence the system (1), is globally asymptotically stable. This completes the proof. □

Remark: Notice that in this theorem, the boundedness of the activation functions are not required. Therefore, the choices of activation functions are much more flexible.

A few previously known results can be obtained from Theorem 3.1.

Corollary 4.0:
(i) [Kaszkurewicz and Bhaya (**10**)] If the interconnection matrix T is diagonally stable, then neural network (1) is globally asymptotically stable for any external inputs I_i and any continuous strictly monotonically increasing activation functions $g_i(u_i)$ on R^1;
(ii) [Forti *et al.* (**11**)] If the interconnection matrix T is symmetric negative semidefinite, then neural network (1) is globally asymptotically stable for any external inputs I_i and any continuous strictly monotonically increasing activation functions $g_i(u_i)$ on R^1.

Proof: (i) Since diagonal stability implies diagonal semistability, the proof is straightforward. (ii) If T is symmetric negative semidefinite, then it is diagonally semistable from the fact that $IT+T^tI = -(-2T)$ where $-2T \geqslant 0$. From Theorem 3.1, the proof can be easily completed. □

Next, we use a few examples to illustrate some applications of Theorem 4.1.

Example 4.1:
Let

$$T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Obviously, $T \in \mathscr{P}_0$. Moreover, $IT+T^tI = 0$, thus, T is diagonally semistable, Theorem 4.1 implies that the network (1) is globally asymptotically stable. However, T is not diagonally stable, therefore, the results in (**10**) and (**11**) can not be applied.

Motivated by this example, the following result is obtained.

Corollary 4.1: If the interconnection matrix T is skew-symmetric, i.e. $T^t = -T$, the neural network (1) is globally asymptotically stable. □

Example 4.2:

$$T = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Since $-T \in \mathscr{P}_0$, from Theorem 3.1, the network (1) is globally asymptotically stable for any $G \in \mathscr{F}^n(R^n; \alpha, \beta)$. However, T is not invertible.

For two-neuron case, the following stronger result is obtained.

Theorem 4.2: For two-neuron case (i.e. $n = 2$), if $-T \in \mathscr{P}$, then the system (1) is globally asymptotically stable for any $G \in \mathscr{F}^2(R^2; \alpha, \beta)$, any capacitances $C_i > 0$, any resistances $R_i > 0$ and any external inputs I_i.

Proof: Let $-T \in \mathscr{P}$, we have $T_{11} > 0$, $T_{22} > 0$ and $T_{11}T_{22} - T_{12}T_{21} > 0$. We want to show that T is diagonally stable. In fact, let $P = \mathrm{diag}\{\lambda_1, \lambda_2\}$ where $\lambda_i > 0$ is to be determined. In order to guarantee $PT + T^tP < 0$, since the diagonal elements of $PT + T^tP$ are negative, it suffices to have $\det(PT + T^tP) > 0$ for two-dimensional case. It is easy to obtain

$$\begin{aligned}\det(PT+T^tP) &= 4\lambda_1\lambda_2 T_{11}T_{22} - (\lambda_1 T_{12} + \lambda_2 T_{21})^2 \\ &= 2\lambda_1\lambda_2(T_{11}T_{22} - T_{12}T_{21}) + \lambda_1(\lambda_2 T_{11}T_{22} - \lambda_1 T_{12}^2) + \lambda_2(\lambda_1 T_{11}T_{22} - \lambda_2 T_{21}^2). \qquad (10)\end{aligned}$$

If $T_{12}T_{21} \geqslant 0$, setting $\lambda_2 T_{11}T_{22} - \lambda_1 T_{12}^2 = 1$ and $\lambda_1 T_{11}T_{22} - \lambda_2 T_{21}^2 = 1$, we have

$$\lambda_1 = \frac{T_{11}T_{22} + T_{21}^2}{(T_{11}T_{22})^2 - (T_{12}T_{21})^2}, \quad \lambda_2 = \frac{T_{11}T_{22} + T_{12}^2}{(T_{11}T_{22})^2 - (T_{12}T_{21})^2}.$$

From $0 \leqslant T_{12}T_{21} < T_{11}T_{22}$, $\lambda_1 > 0$ and $\lambda_2 > 0$. Taking this into Eq. (10), we obtained $\det(PT + T^tP) = 2\lambda_1\lambda_2(T_{11}T_{22} - T_{12}T_{21}) + \lambda_1 + \lambda_2 > 0$. If $T_{12}T_{21} < 0$, we choose such positive number λ_1 and λ_2 that $\lambda_1|T_{12}| = \lambda_2|T_{21}|$, thus $\lambda_1 T_{12} + \lambda_2 T_{21} = 0$. Taking this into Eq. (10), we have $\det(PT + T^tP) = 4\lambda_1\lambda_2 T_{11}T_{22} > 0$. Therefore, T is diagonally stable. From Theorem 4.1, the system (1) is globally asymptotically stable. □

We conjecture that when $-T \in \mathscr{P}$ and T is stable, then the system (1) is also globally asymptotically stable. However, we have not been able to prove or disprove it yet.

Theorem 4.1 can be extended to apply to a special class of large-scale neural networks. suppose that T (upper) block triangular form:

$$T = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1r} \\ 0 & A_{22} & \dots & A_{2r} \\ \vdots & \dots & \ddots & \vdots \\ 0 & 0 & \dots & A_{rr} \end{bmatrix}$$

where A_{ij} is an $n_i \times n_i$ matrix ($i = 1, 2, \dots, r$). We first give the following property of this block matrix.

Lemma 4.1: (15) Let T be an upper block triangular matrix in the above form with square diagonal blocks $A_{11}, \dots, A_{rr}$. Then T is diagonally semistable if and only if for $i < j$, the matrix

$$\begin{pmatrix} A_{ii} & A_{ij} \\ 0 & A_{jj} \end{pmatrix}$$

is diagonally semistable.

Applying this lemma and Theorem 4.1, we obtain the following result.

Theorem 4.3: Suppose that the interconnection matrix T can be written as a triangular block matrix form with square diagonal blocks $A_{11}, \dots, A_{rr}$. If the matrix

$$\begin{pmatrix} A_{ii} & A_{ij} \\ 0 & A_{jj} \end{pmatrix} \quad (1 \leqslant i < j \leqslant n)$$

for upper triangular case or

$$\begin{pmatrix} A_{ii} & 0 \\ A_{ji} & A_{jj} \end{pmatrix} \quad (1 \leqslant i < j \leqslant n)$$

for lower triangular case are diagonally semistable, then the network (1) is globally asymptotically stable for any $G \in \mathscr{F}^n(R^{n}, \alpha, \beta)$. □

Using this theorem, global stability of high dimensional neural network (1) can be reduced to the global stability of some smaller networks. One interesting special application of this result is to solve the global stability of triangular neural networks. These networks have been found applications in A/D conversion or quantization area (**3**, **17**).

Corollary 4.2: For the network (1) with continuous strictly monotonically increasing activation functions, if T is in a triangular form with negative diagonal elements, i.e. $T_{ii} < 0$, then it is globally asymptotically stable.

Proof: We only prove the lower triangular case, i.e. $T_i = 0$ if $i < j$. From Theorem 4.3, it only suffices to prove that for any $i > j$, the matrix

$$\begin{pmatrix} T_{ii} & 0 \\ T_{ij} & T_{jj} \end{pmatrix}$$

is diagonally stable. From Theorem 4.2, it suffices to show that

$$\begin{pmatrix} -T_{ii} & 0 \\ -T_{ij} & -T_{jj} \end{pmatrix} \in \mathscr{P},$$

this is obvious because $T_{ii} < 0$ for any $i = 1, \ldots, n$. This completes the proof. □

In the above, the sufficient conditions for global stability we developed are independent of the shapes of activation functions, C_i, R_i and I_i. This may be conservative for certain applications. The next result is the relaxed version of Theorem 4.1.

Theorem 4.4: Suppose that the activation functions $g_i(x_i)$ are bounded continuous and strictly monotonically increasing ones which have the following property:

$$0 < \frac{g_i(x) - g_i(y)}{x - y} \leqslant \omega_i \quad \text{for any } x, y \in R^1. \tag{11}$$

Let $\Omega = \text{diag}\{\omega_1, \ldots, \omega_n\}$. If the matrix $R\Omega T - I$ is diagonally stable, then the neural network (1) is globally asymptotically stable for any capacitance C_i and any external input I_i.

Proof: As in the proof of Theorem 3.2, applying Schauder's Fixed Point Theorem (**13**), we know that there exists at least one equilibrium point, say, $u^* = (u_1^*, \ldots, u_n^*)^t$. Following a similar procedure in the proof of Theorem 4.1, we obtain (9). Because $\phi_i(x_i)$ is strictly monotonically increasing, from (11), we can easily deduce

$$\omega_i x_i \phi_i(x_i) \geqslant \phi_i^2(x_i). \tag{12}$$

Let $\Pi(CR)^{-1} = \mathrm{diag}\{r_1, \ldots, r_n\}$, which is a positive diagonal matrix, from (12) we have

$$\Phi^t(x)\Pi(CR)^{-1}x = \sum_{i=1}^{n} r_i\phi_i(x_i)x_i \geqslant \sum_{i=1}^{n} r_i\omega_i^{-1}\phi_i^2(x_i)$$

$$= \Phi^t(x)\Pi(CR)^{-1}\Omega^{-1}\Phi(x).$$

Taking this into (9), we obtain the following:

$$\dot{V}(x) \leqslant \Phi^t(x)[\Pi(C^{-1}(T-(R\Omega)^{-1}) + [C^{-1}(T-(R\Omega)^{-1})]^t\Pi]\Phi(x). \tag{13}$$

Suppose that $R\Omega T - I$ is diagonally stable, and $R\Omega$ is a positive diagonal matrix, $T-(R\Omega)^{-1}$ is also diagonally stable. So there exists a positive diagonal matrix P and a positive matrix Q such that $P(T-(R\Omega)^{-1}) + (T-(R\Omega)^{-1})^t P = -Q$. From this and (13) by letting $\Pi = PC$, we obtain

$$\dot{V}(x) \leqslant -\Phi^t(x)Q\Phi(x) \leqslant -\lambda_{\min}(Q)\Phi^t(x)\Phi(x) < 0$$

for any $x \neq 0$ due to the fact that $\phi_i(x_i)$ is strictly monotonically increasing. Also $V(x)$ is radically unbounded, from Lyapunov stability theorem (**16**), the system (1) is globally asymptotically stable. This completes the proof. □

Remarks:

(1) If $R_i = \omega_i = C_i = 1$ and $g_i(x_i)$ is continuously differentiable, Theorem 4.4 reduces to the main result in (**9**).

(2) If T is diagonally semistable, then there exists a positive diagonal matrix P such that $PT + T^tP \leqslant 0$, therefore,

$$P(T-(R\Omega)^{-1} + (T-(R\Omega)^{-1})^t P = PT + T^tP - 2(R\Omega)^{-1} < 0,$$

this means that $T-(R\Omega)^{-1}$ is diagonally stable, so is $R\Omega T - I$. So Theorem 4.4 implies Theorem 4.1 for the case when the slopes of activation functions have the bounds (11).

(3) Notice that the sufficient condition in this theorem is also independent of the capacitances and external inputs.

Applying Theorem 4.2 and Theorem 4.4, we can obtain the following simpler testable result for two-neuron case.

Corollary 4.3: For the two-neuron network (1) [i.e. $n = 2$] as in Theorem 4.4, if $I - R\Omega T \in \mathscr{P}$, then the network (1) is globally asymptotically stable.

Proof: From the proof of Theorem 4.2, if $I - R\Omega T \in \mathscr{P}$, then $R\Omega T - I$ is diagonally stable. From theorem 4.4, the neural network (1) is globally asymptotically stable. □

A similar result to Theorem 4.3 can be formalized for Theorem 4.4. We have:

Theorem 4.5: The assumptions about the network (1) are as in Theorem 4.4. Let T be in a lower triangular block matrix form, say, $T = (A_{ij})$ where A_{ij} is an $n_i \times n_j$ matrix for $1 \leqslant i,j \leqslant r$, $A_{ij} = 0$ for all $i < j$. Let $(R\Omega)^{-1} = (E_{ij})$ where E_{ij} is also $n_i \times n_j$ matrix,

$E_{ij} \neq 0$ if $i \in j$ and E_{ii} is also a diagonal submatrix of $(R\Omega)^{-1}$. If for any $1 \leqslant i < j \leqslant r$, the matrix

$$\begin{pmatrix} A_{ii} - E_i & 0 \\ A_{ji} & A_{jj} - E_{jj} \end{pmatrix}$$

is diagonally stable, then the system (1) is globally asymptotically stable. A similar result is true for the upper triangular case.

Proof: This can be proved by the following fact: For lower triangular matrix $X = (X_{ij})$ where X_{ij} is an $n_i \times n_j$ matrix and $X_{ij} = 0$ for $i < j$, if for any $i < j$,

$$\begin{pmatrix} X_{ii} & 0 \\ X_{ji} & X_{jj} \end{pmatrix}$$

is diagonally stable, then X is also diagonally stable. In fact, if

$$\begin{pmatrix} X_{ii} & 0 \\ X_{ji} & X_{ji} \end{pmatrix}$$

is diagonally stable, then for sufficiently small

$$\varepsilon > 0, \quad \begin{pmatrix} X_{ii} & 0 \\ X_{ji} & X_{jj} \end{pmatrix} + \varepsilon I$$

is diagonally stable, and is also diagonally semistable. From Lemma 4.1, $X + \varepsilon I$ is diagonally semistable, i.e. there exists a positive diagonal matrix P and a positive semidefinite matrix Q such that $P(X + \varepsilon I) + (X + \varepsilon I)^{\mathrm{t}} = -Q$, so $PX + X^{\mathrm{t}}P = -(Q + 2\varepsilon I)$. Since $Q + 2\varepsilon P$ is positive definite, X is diagonally stable (from this procedure, we can see that when the "semistable" is replaced with "stable" in Lemma 4.1, the result is still valid). Applying this fact to matrix $T - (R\Omega)^{-1}$, we can easily complete the proof. □

From this result, Corollary 4.2 can be generalized in the following:

Corollary 4.4: for the network (1) satisfying the assumptions in Theorem 4.4, if T is in triangular form with nonpositive diagonal elements, i.e. $T_{ii} \leqslant 0$ $(i = 1, \ldots, n)$, then it is globally asymptotically stable.

Proof: Notice that $T - (R\Omega)^{-1}$ is also in the triangular form with negative diagonal elements, then the proof is straightforward. □

Remark: In **(17)**, the result for $T_{ii} = 0$ was obtained, which is a special case of Corollary 4.4. Michel and Gray **(3)** obtained a generically similar result, but for local asymptotic stability.

An $n \times n$ matrix $A = (a_{ij})$ is said to be a H-matrix if the matrix $M(A)$ defined by

$$[M(A)]_{ij} = \begin{cases} |a_{ii}|, & i = j, \\ -|a_{ij}|, & i \neq j, \end{cases}$$

is an M-matrix, i.e. the eigenvalues of $M(A)$ have positive real parts. It is known **(15)** that for a H-matrix A, A is diagonally stable if and only if A is nonsingular and the diagonal elements of A are negative. From this and the fact that $M(A)$ is an M-matrix if and only if $M(A) \in \mathscr{P}$ (**18**), the following is straightforward.

Corollary 4.5: For the network (1) with bounded continuous strictly monotonically increasing functions $g_i(x_i)$ satisfying (11), define $H = (H_{ij})$ where

$$H_{ij} = \begin{cases} |T_{ii} - 1/(R_i\omega_i)|, & i = j, \\ -|T_{ij}|, & i \neq j. \end{cases}$$

If $H \in \mathscr{P}$, $\det(I - R\Omega T) \neq 0$ and $T_{ii} < 1/(R_i\omega_i)$ $(i = 1, 2, \ldots, n)$, then the network (1) is globally asymptotically stable. □

Remark: Using M-matrix property (**18**), many Gershgorin-type sufficient conditions (**7, 9**) can be derived from this corollary. This approach has been intensively investigated by Michel *et al.* (**3**, **4**) who obtained a similar result for local asymptotic stability, however, Corollary 4.4 claims a similar condition for global asymptotic stability.

Example 4.3: Consider the example in Hopfield and Tank (**1**) which is also studied in Michel *et al.* (**6**) for multiple equilibria case. For this model:

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad g_i(x_i) = \frac{2}{\pi}\tan^{-1}\left(\frac{\lambda\pi}{2}x_i\right),$$

where $\lambda > 0$. Let $C_i = R_i = 1$ and I_i is any constant, the network (1) becomes

$$\begin{aligned} \dot{x}_1 &= -x_1 + g_2(x_2) + I_1 \\ \dot{x}_2 &= -x_2 + g_1(x_1) + I_2. \end{aligned} \tag{14}$$

Since $\max_{x_i \in R^1} \dot{g}_i(x_i) = \lambda$, the matrix $\Omega = \lambda I$. Applying Corollary 4.3, if

$$I - R\Omega T = \begin{pmatrix} 1 & -\lambda \\ -\lambda & 1 \end{pmatrix}$$

is $\mathscr{P}$ matrix, i.e. $0 < \lambda < 1$, then the network (1) is globally asymptotically stable. In fact, for this model, the range for λ is the best we can hope for global asymptotic stability. For illustration, we choose $I_1 = I_2 = 0$, the origin $(0, 0)^t$ is one equilibrium point, the linearized system matrix is

$$\begin{pmatrix} -1 & \lambda \\ \lambda & -1 \end{pmatrix}$$

which is not asymptotically stable for $\lambda \geqslant 1$, therefore the system (1) will never be globally asymptotically stable. Notice that, however, $-T$ itself is not a $\mathscr{P}_0$ matrix, Theorem 4.1 can not be applied.

As we noted in the last section that when T is skew-symmetric, the equilibrium of (1) is unique. For the stability of this equilibrium, we have the following result.

Corollary 4.6: For the network (1) with bounded continuous strictly monotonically increasing activation functions $g_i(x_i)$ satisfying (11), if T is skew-symmetric, i.e. $T^t = -T$, then it is globally asymptotically stable.

Proof: Since

$$I(T-(R\Omega)^{-1})+(T-(R\Omega)^{-1})^{t}I = T+T^{t}-2(R\Omega)^{-1} = -2(R\Omega)^{-1} < 0,$$

we obtain that $T-(R\Omega)^{-1}$ is diagonally stable, so is $R\Omega T-I$. From Theorem 4.4, the network (1) is globally asymptotically stable. □

V. Conclusions

In this paper, we have developed a set of sufficient and "almost necessary" conditions for the existence and uniqueness of an equilibrium point for a class of asymmetrical Hopfield-type neural networks, and provided some new sufficient conditions for the global asymptotic stability of the networks. Future research will be directed to the applications of these global results to the practical problems such as the nonlinear optimizations and Winner-Take-All circuit designs. The results are perfect for a neural circuit running in real time and resetting is not necessary for changing external stimulating inputs.

References

(1) J. J. Hopfield and D. W. Tank, "Neurons with graded response have collective computational properties like those of two-state neurons," *Proceedings of the National Academy of Sciences*, Vol. 79, pp. 3088–3092, 1984.

(2) J. J. Hopfield and D. W. Tank, "Computing with neural circuits: a model," *Sciences*, Vol. 233, pp. 625–633, 1986.

(3) A. N. Michel and D. L. Gray, "Analysis and synthesis of neural networks with lower block triangular interconnecting structure," *IEEE Trans. Circuits and Systems*, Vol. 37, No. 10, pp. 1267–1283, 1990.

(4) J. H. Li, A. N. Michel and W. Porod, "Qualitative analysis and synthesis of a class of neural networks," *IEEE Trans. Circuits and Systems*, Vol. 35, No. 8, pp. 976–985, 1988.

(5) L. T. Grujic and A. N. Michel, "Exponential stability and trajectory bounds of neural networks under structural variations," *IEEE Trans. Circuits and Systems*, Vol. 38, No. 10, pp. 1182–1192, 1991.

(6) A. N. Michel, J. A. Farrel and W. Porod, "Qualitative analysis of neural networks," *IEEE Trans. Circuits and Systems*, Vol. 36, No. 2, pp. 229–243, 1989.

(7) M. W. Hirsch, "Convergent activation dynamics in continuous time networks," *Neural Networks*, Vol. 2, pp. 331–349, 1989.

(8) D. G. Kelly, "Stability in contractive nonlinear neural networks," *IEEE Trans. Biomedical Engr.*, Vol. 3, No. 3, pp. 231–242, 1990.

(9) K. Matsuoka, "Stability conditions for nonlinear continuous neural networks with asymmetric connection weights," *Neural Networks*, Vol. 5, pp. 495–500, 1992.

(10) E. Kaszkurewicz and A. Bhaya, "On a class of globally state neural circuits," *IEEE Trans. Circuits and Systems—I: Fundamental Theory and Applications*, Vol. 41, No. 2, pp. 171–174, 1994.

(11) M. Forti, S. Manetti and M. Marini, "Necessary and sufficient conditions for absolute stability of neural networks," *IEEE Trans. Circuits and Systems—I: Fundamental Theory and Applications*, Vol. 41, No. 7, pp. 491–494, 1994.

(12) I. W. Sandberg and A. N. Wilson, "Some network-theoretic properties of nonlinear dc transistor networks," *Bell Syst. Tech. J.*, Vol. 48, pp. 1293–1311, 1969.

(13) L. V. Kantorovich and G. P. Akilov, "Functional Analysis in Normed Spaces," p. 640, The MacMillan Company, New York, 1964.
(14) L. O. Chua and L. Yang, "Cellular neural networks: theory," *IEEE Trans. Circuits and Systems*, Vol. 35, No. 10, pp. 1257–1272, 1988.
(15) D. Hershkowitz, "Recent directions in matrix stability," *Linear Algebra and its Applications*, Vol. 171, pp. 161–186, 1992.
(16) J. A. La Salle and S. Lefschetz, "Stability by Liapunov's Direct Method with Applications," Academic Press, New York, 1961.
(17) G. Avitabile, M. Forti and M. Marini, "On a class of nonsymmetrical neural networks with application to ADC," *IEEE Trans. circuits and Systems*, Vol. 38, No. 2, pp. 202–209, 1991.
(18) A. Berman and R. J. Plemmons, "Nonnegative Matrices in the Mathematical Sciences," Academic Press, New York, 1979.
(19) A. Guez, V. Protopopsecu and J. Barhen, "On the stability, storage capacity, and design of nonlinear continuous neural networks," *IEEE Trans. Systems, Man and Cybernetics*, Vol. 18, No. 1, pp. 80–87, 1988.