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Almost sure and δ-moment stability of jump linear systems

YUGUANG FANG†‡, KENNETH A. LOPARO† and XIANGBO FENG†

In this paper, we study the almost sure-stability and δ-moment stability of discrete-time jump linear systems with a finite state form process. Criteria for almost sure and δ-moment stability, which are necessary and sufficient conditions for (scalar) one-dimensional systems, are derived. The relationship between almost sure and δ-moment stability for (scalar) one-dimensional systems is also presented. If the system matrices commute, necessary and sufficient conditions for almost sure stability are obtained.

1. Introduction

Jump linear systems are a family of stochastic systems of the form

$$x_{t+1} = H(\sigma_t)x_t + G(\sigma_t)u_t,\ t \in Z^+ = \{0, 1, \dots\} \tag{1.1}$$

for the discrete-time case, or

$$\dot{x}_t = H(\sigma_t)x_t + G(\sigma_t)u_t,\ t \in \mathcal{R}^+ = [0, +\infty) \tag{1.2}$$

for the continuous-time case. The finite-state random process $\{\sigma_t\}$ is usually a Markov jump process, referred to as the form process and models the possible changes in system structure. The system models (1.1) and (1.2) are often encountered in engineering applications; for example, in the analysis of control systems subject to abrupt phenomena such as component and interconnection failure (Chizeck *et al.* 1986, Mariton 1990), and more recently in the study of systems with random communication delays (Krtolica *et al.* 1991, Fang *et al.* 1991). A significant effort has been devoted to the optimal control of jump linear systems with a quadratic cost functional and to developing notions of controllability and observability for this class of systems. Many important results concerning the analysis and design of such systems have been obtained, see, for example, Ji and Chizeck (1990), Mariton (1990) and the references cited therein.

It is common knowledge that the stability of a dynamical system is one of the primary concerns in the design and synthesis of a control system. The study of stability of jump linear systems has attracted the attention of many researchers. The earliest work can be traced back to Rosenbloom (1954), who was interested in stability properties of the moments of such systems. Bellman (1960) was the first to study the moment stability of (1.1) with an independent and identically distributed (i.i.d.) form process using the Kronecker matrix product. Bergen (1960) used a similar idea to study the moment stability properties of continuous time systems (1.2) where $\{\sigma_t\}$ is a random process with piecewise constant

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† Department of Systems Engineering, Case Western Reserve University, Cleveland, OH 44106, U.S.A.

‡ On leave from The Institute of Automation, Qufu Normal University, Qufu, Shandong, the People's Republic of China.

sample paths. Later, Bhuracha (1961) used the idea developed by Bellman (1960) to generalize Bergen's results and studied both the asymptotic stability and the exponential stability of the mean. Darkhovskii and Leibovich (1971) investigated the second moment stability of systems of the form (1.2) where the time intervals between jumps are i.i.d. and the mode process for the system is a finite state Markov chain with a stationary probability transition matrix. In this work, they obtained necessary and sufficient conditions for second moment stability in terms of the Kronecker matrix product. This is an extension of Bhuracha's result. Kats and Krasovskii (1960) and Bertram and Sarachik (1959) used a stochastic version of Lyapunov's second method to study almost sure and moment stability. Unfortunately, Lyapunov functions are, in general, difficult to construct and this is a recognized disadvantage of Lyapunov's second method. Moreover, for most cases, the criteria obtained from this method are similar to the moment stability criteria, which are usually too conservative to be of practical value. See also the work of Mariton (1989).

Recently, Ji *et al.* (1991) Feng (1990) and Feng *et al.* (1992) used Lyapunov's second method to study the stability of (1.1) or (1.2) with a finite state Markov chain form process, and obtained necessary and sufficient conditions for the second moment stability both for the discrete time case (1.1) and for the continuous time case (1.2), respectively.

As Kozin (1969) pointed out, moment stability implies almost sure stability under fairly general conditions, but the converse is not true. This is a unique aspect of stochastic stability problems—in deterministic systems, if the sample path is stable, then all moments are certainly stable. In applications, almost sure (sample path) stability is the desirable property, because sample path behaviour can be observed in practice. It often turns out that second moment stability criteria, which are commonly associated with the synthesis of optimal closed-loop systems derived by minimizing a quadratic cost functional, are too conservative to be useful in practical applications. Several illustrative examples were given by Mitchell and Kozin (1974). This is a motivation for this work, where we seek to obtain more general stability criteria for almost sure (sample path) stability. The recent development of the Lyapunov exponent method offers potential for obtaining testable necessary and sufficient conditions for almost sure stability (Loparo and Blankenship 1984, Feng *et al.* 1992, Ji *et al.* 1991, Mariton 1988). For a comprehensive and mathematically oriented treatment of Lyapunov exponents, the reader is encouraged to refer to Arnold and Wihstutz (1986), Bougerol and Lacroix (1985). Although it is known that the (top) Lyapunov exponent is naturally related to sample path stability, it is usually very difficult to compute analytically, estimate or determine the sign of the top Lyapunov exponent. More advanced results, in particular those related to the development of computational schemes, are needed before this approach can be applied to the analysis and design of control systems.

In the spirit of the above works, there are many unsolved problems for the stability of jump linear systems, most notably, deriving testable conditions which yield a reasonable estimate of the almost-sure stability region in the space of system parameters. It is known that for some special systems, the almost-sure stability property of the system is directly related to δ-moment stability (with small δ). In this case, a testable condition for δ-moment stability will give a reasonable estimate of the almost-sure stability domain. Although some results and conjectures exist for scalar jump linear systems (Feng *et al*. 1992) and for stochastic systems with a special structure (Arnold 1982), δ-moment stability conditions and their relationship to almost-sure stability remains an open question and presents an active research direction.

In this paper, we present some basic results on almost-sure and δ-moment stability for jump linear systems. We concentrate on the family of discrete-time systems with an i.i.d. form process. This is the simplest model with which one can deal. However, most of the results reported here can be directly extended to the situation where the form process is a general Markovian process both in continuous and discrete time. From this aspect, the results are illustrative. The approach we adopt in this work is straightforward, and only relies on some basic results from matrix algebra and probability theory. The paper is organized as follows: in § 2, we study the almost-sure, second moment and mean stability of the system (1.1). Some testable sufficient conditions are obtained. In § 3, the δ-moment ($\delta > 0$) stability is investigated and its relationship to almost-sure stability for scalar systems is also addressed. Section 4 is devoted to systems with special commuting structures, and necessary and sufficient conditions for various types of stability properties are obtained. Finally, we present concluding remarks and possible generalizations of the results presented in this paper in § 5.

2. Almost sure, second moment and mean stability

Consider the discrete-time jump linear system given by

$$x_{t+1} = H(\sigma_t)x_t, \; t \in \mathscr{Z}^+ = \{0, 1, \ldots\} \tag{2.1}$$

where $\sigma_t \in \{1, 2, \ldots, N\}$ is assumed to be an i.i.d. random sequence with the common probability distribution $P\{\sigma_t = j\} = p_j$ for $j = 1, 2, \ldots, N$, unless stated otherwise. Without loss of generality, assume that $p_j \neq 0$ and $x_0 \in \mathscr{R}^n$ is a non-random constant. The following notation will be used throughout the remainder of the paper. Let $\underline{N} \stackrel{\text{def}}{=} \{1, 2, \ldots, N\}$ and $\Delta_i = \det(H(i))$ for $i \in \underline{N}$. Let $\lambda(A)$ denote the collection of all eigenvalues of the matrix A, i.e. the spectrum of A, and $\rho(A)$ denote the spectral radius of A, i.e. $\rho(A) = \max\{|\lambda| : \lambda \in \lambda(A)\}$. Let $\mu_i = \rho(H(i))$ be the spectral radius of $H(i)$ for $i \in \underline{N}$. If A is a positive semi-definite matrix, let $\lambda_{\max}(A) = \max\{\lambda : \lambda \in \lambda(A)\}$ denote the largest eigenvalue of A, and let $\lambda_i = \lambda_{\max}(H'(i)H(i))$ for $i \in \underline{N}$, where $'$ denotes matrix transpose. Let $I_i(x)$ be the indicator function which is defined as

$$I_i(x) = \begin{cases} 1, & \text{if } x = i \\ 0, & \text{otherwise} \end{cases}$$

Finally, for $(\eta_1, \ldots, \eta_N)$ given with $\eta_i \geq 0$, define the set of matrices

$$\mathscr{H}(\eta_1, \eta_2, \ldots, \eta_N) = \{(A_1, A_2, \ldots, A_N) : A_i \in \mathscr{R}^{n \times n}, \lambda_{\max}(A_i'A_i) \leq \eta_i, \; \forall i \in \underline{N}\} \tag{2.2}$$

The following stability concepts are important.

Definition 2.1: The jump linear system (2.1) is said to be

(i) *almost surely stable*, if for any $x_0 \in \mathscr{R}^n$, $\lim_{t \to +\infty} \|x_t\| = 0$ almost surely (with probability one).

(ii) *δ-moment stable ($\delta > 0$)*, if for any $x_0 \in \mathcal{R}^n$, $\lim_{t\to+\infty} E\{\|x_t\|^\delta\} = 0$.

(iii) *mean stable*, if for any $x_0 \in \mathcal{R}^n$, $\lim_{t\to+\infty} E\{x_t\} = 0$.

(iv) *almost surely $\mathcal{H}(\eta_1, \ldots, \eta_N)$ stable*, if for any $(H(1), \ldots, H(N)) \in \mathcal{H}(\eta_1, \ldots, \eta_N)$, the system defined by (2.1) is almost surely stable.

If $\{\sigma_k\}$ is a finite state Markov chain with initial distribution π_0, we say the system is stable in any one of the above senses, if the corresponding property holds not only for any $x_0 \in \mathcal{R}^n$, but also for any initial distribution π_0 of $\{\sigma_k\}$.

□

Note that mean stability is different from first moment stability (because the $\|\cdot\|$ is used in the definition of first moment stability) and almost-sure $\mathcal{H}(\lambda_1, \ldots, \lambda_N)$ stability (iv) can be interpreted as a type of robust almost-sure stability concept for stochastic systems.

In this section, we present some testable conditions for the various types of stability of (2.1) given in Definition 2.1. These conditions are easy to test and for certain classes of systems, they are necessary and sufficient. Our results are also valid with σ_t, a finite-state Markov chain, although we only present the details for the i.i.d. case in this paper. For more detailed results concerning the Markovian case, see Fang *et al*. (1991). The following theorem is one of our main results.

Theorem 2.1—Almost-sure stability properties: *For the jump linear system* (2.1), *we have*

(1) *a sufficient condition for almost-sure stability is*

$$\lambda_1^{p_1}\lambda_2^{p_2} \ldots \lambda_N^{p_N} < 1 \tag{2.3}$$

in particular, for a scalar system (the dimension of the state space $n = 1$), the above condition is also necessary

(2) if $\Delta_1^{p_1}\Delta_2^{p_2} \ldots \Delta_N^{p_N} \geq 1$, *then for almost all (with respect to Lebesque measure) $x_0 \in \mathcal{R}^n$, $\lim_{t\to+\infty} \|x_t\| \neq 0$ and thus, the system is not almost surely stable (almost sure unstable)*

(3) *necessary and sufficient condition for the system to be almost surely $\mathcal{H}(\eta_1, \ldots, \eta_N)$ stable is*

$$\eta_1^{p_1}\eta_2^{p_2} \ldots \eta_N^{p_N} < 1 \tag{2.4}$$

Before we prove Theorem 2.1, we make some comments about the results. Note that $\sqrt{\lambda_i}$ is the largest singular value of $H(i)$ and Condition (1) (equation (2.3)) is only a sufficient condition for a.s. (almost sure) stability. Condition (2) implies that $\Delta_1^{p_1}\Delta_2^{p_2} \ldots \Delta_N^{p_N} < 1$ is a necessary condition for a.s. stability. Condition (3) is a necessary and sufficient condition for robust a.s. stability. Another interpretation of (3) is that the condition (2.3) becomes necessary and sufficient for a.s. $\mathcal{H}(\lambda_1, \ldots, \lambda_N)$ stability. All these conditions are easily testable and examples are presented after the proof of the theorem to illustrate the range of applicability of these results. To prove the theorem, we need the following lemma which is a fundamental result of stochastic processes and its proof can be found in, say, Shiryayev (1984).

Lemma 2.1—Law of the iterative logarithm: *Let $\{\xi_k\}$ be a sequence of independent identically distributed random variables with $E\xi_k = 0$ and*

$E\xi_k^2 = \sigma^2 < +\infty$, *then*

$$P\left(\overline{\lim_{n\to\infty}} \frac{\sum_{i=1}^{n} \xi_i}{\sqrt{2\sigma^2 n \log\log n}} = 1\right) = 1$$

$$P\left(\underline{\lim_{n\to\infty}} \frac{\sum_{i=1}^{n} \xi_i}{\sqrt{2\sigma^2 n \log\log n}} = -1\right) = 1$$

Proof of Theorem 2.1: For any k and x, it is easy to show that

$$x'H'(\sigma_k)H(\sigma_k)x \leqslant \lambda_1^{I_1(\sigma_k)}\lambda_2^{I_2(\sigma_k)} \ldots \lambda_N^{I_N(\sigma_k)}x'x \tag{2.5}$$

Thus, we have

$$\begin{aligned} x_k'x_k &= x_0'H'(\sigma_0)H'(\sigma_1) \ldots H'(\sigma_{k-1})H(\sigma_{k-1}) \ldots H(\sigma_0)x_0 \\ &\leqslant \lambda_1^{I_1(\sigma_{k-1})} \ldots \lambda_N^{I_N(\sigma_{k-1})}x_0'H'(\sigma_0) \ldots H'(\sigma_{k-2})H(\sigma_{k-2}) \ldots H(\sigma_0)x_0 \\ &\leqslant \cdots \leqslant \lambda_1^{\sum_{i=0}^{k-1}I_1(\sigma_i)}\lambda_2^{\sum_{i=0}^{k-1}I_2(\sigma_i)} \ldots \lambda_N^{\sum_{i=0}^{k-1}I_N(\sigma_i)}x_0'x_0 \\ &= (\lambda_1^{\frac{1}{k}\sum_{i=0}^{k-1}I_1(\sigma_i)} \ldots \lambda_N^{\frac{1}{k}\sum_{i=0}^{k-1}I_N(\sigma_i)})^k x_0'x_0 \end{aligned} \tag{2.6}$$

Since $\{\sigma_k\}$ is an i.i.d. sequence, for each $j \in \underline{N}$, $\{I_j(\sigma_k)\}$ is also an i.i.d. sequence. Therefore, from the Law of Large Numbers (Shiryayev, 1984, p. 366), we have

$$\lim_{k\to\infty} \frac{1}{k}\sum_{i=0}^{k-1} I_j(\sigma_i) = E\{I_j(\sigma_i)\} = p_j \quad \text{a.s.} \quad \forall j \in \underline{N}$$

It follows that

$$\lim_{k\to\infty} \lambda_1^{\frac{1}{k}\sum_{i=0}^{k-1}I_1(\sigma_i)}\lambda_2^{\frac{1}{k}\sum_{i=0}^{k-1}I_2(\sigma_i)} \ldots \lambda_N^{\frac{1}{k}\sum_{i=0}^{k-1}I_N(\sigma_i)} = \lambda_1^{p_1}\lambda_2^{p_2} \ldots \lambda_N^{p_N} \quad \text{a.s.}$$

From (2.3), the right-hand side of the above inequality is less than 1, almost surely, and there exist a $k_0 = k_0(\omega)$ and $\lambda < 1$, so that

$$x_k'x_k \leqslant \lambda^k x_0'x_0, \quad \forall k \geqslant k_0 \quad \text{a.s.} \tag{2.7}$$

From this we conclude that the system (2.1) is almost surely (exponentially) stable. This establishes the first statement of (1).

Next, we apply Lemma 2.1 to show that (2.3) is also necessary when $n = 1$. In this case, all the inequalities in (2.5)–(2.6) are actually equalities. It is easy to see that if $\lambda_1^{p_1}\lambda_2^{p_2} \ldots \lambda_N^{p_N} > 1$, we have $\lim_{n\to\infty} x_k^2 = +\infty$ a.s. for any $x_0 \neq 0$, which contradicts the almost-sure stability. Suppose that $\lambda_1^{p_1}\lambda_2^{p_2} \ldots \lambda_N^{p_N} = 1$. Then, we have $\sum_{i=1}^{N} p_i \log \lambda_i = 0$ and also

$$\begin{aligned} x_k^2 &= \lambda_1^{\sum_{i=0}^{k-1}I_1(\sigma_i)} \ldots \lambda_N^{\sum_{i=0}^{k-1}I_N(\sigma_i)}x_0^2 \\ &= \exp\left(\sum_{i=0}^{k-1}\sum_{j=1}^{N} I_j(\sigma_i)\log\lambda_j\right) x_0^2 \end{aligned}$$

Let $\xi_i = \sum_{j=1}^{N} I_j(\sigma_i)\log\lambda_j$, with $\{\sigma_i\}$ an i.i.d. sequence, $\{\xi_i\}$ is also an i.i.d.

sequence. Thus, we obtain

$$E\xi_i = \sum_{j=1}^{N}(\log\lambda_j)EI_j(\sigma_i) = \sum_{j=1}^{N}(\log\lambda_j)p_j = 0$$

and

$$E\xi_i^2 = E\left(\sum_{j=1}^{N}(\log\lambda_j)I_j(\sigma_i)\right)^2 = \sum_{j=1}^{N}p_j(\log\lambda_j)^2 < +\infty$$

From Lemma 2.1, we obtain that

$$\overline{\lim_{k\to\infty}}\frac{\sum_{i=0}^{k-1}\xi_i}{\sqrt{2\sigma^2\log\log n}} = 1 \quad \text{a.s.}$$

Thus, $\sum_{i=0}^{k-1}\xi_i$ is unbounded from above almost surely and so is x_k^2. This contradicts the almost-sure stability assumption and completes the proof of (1).

For (3), the sufficiency directly follows from (1) and necessity is proved next. Take $H(j) = \sqrt{\eta_j}I$ for $j \in \underline{N}$, where I denotes the identity matrix. Then $(H(1), \ldots, H(N)) \in \mathcal{H}(\eta_1, \ldots, \eta_N)$ and

$$x_k'x_k = \eta_1^{\sum_{i=0}^{k-1}I_1(\sigma_i)}\ldots\eta_N^{\sum_{i=0}^{k-1}I_N(\sigma_i)}x_0'x_0$$

Using the same arguments as in the proof of the necessity of (2.3) for the scalar case, we see that a.s. stability of (2.1) with $H(j) = \sqrt{\eta_j}I$ implies

$$\eta_1^{p_1}\eta_2^{p_2}\ldots\eta_N^{p_N} < 1$$

This completes the proof of (3).

Finally, we show (2) by contradiction. Suppose that $\lim_{k\to+\infty}x_k = 0$ a.s. for all $x_0 \in S \subset \mathcal{R}^n$ where S is a subset of $\mathcal{R}^n$ which has positive Lebesgue measure. Then, there is a basis $(v_1, \ldots, v_n\}$ of $\mathcal{R}^n$ so that for each j

$$\lim_{k\to+\infty}H(\sigma_k)H(\sigma_{k-1})\ldots H(\sigma_0)v_j = 0, \quad \text{a.s.}$$

It follows that

$$\lim_{k\to+\infty}H(\sigma_k)H(\sigma_{k-1})\ldots H(\sigma_0)[v_1, \ldots, v_n] = 0 \in \mathcal{R}^{n\times n} \quad \text{a.s.}$$

and therefore

$$\begin{aligned}\lim_{k\to+\infty}\det(H(s_k)H(\sigma_{k-1})\ldots H(\sigma_0)) &= \lim_{k\to+\infty}(\Delta_1^{\frac{1}{k}\sum_{i=0}^{k}I_1(\sigma_i)}\ldots\Delta_N^{\frac{1}{k}\sum_{i=0}^{k}I_N(\sigma_i)})^k\\ &= 0 \quad \text{a.s.}\end{aligned}$$

Using a similar argument as in the proof of the necessity of (2.3) for the scalar case (using Lemma 2.1 and the law of large numbers), we conclude that

$$\Delta_1^{p_1}\ldots\Delta_n^{p_N} < 1$$

This is the desired contradiction. □

Before presenting an example to illustrate the results, further observations are in order. First, from Lemma 2.1, we can also conclude that the sequence $\sum_{i=0}^{k-1}\xi_i$ is not bounded from below, hence there exists a subsequence $\{n_k\}$ so that $x_{n_k}^2 \to 0$ $(k \to \infty)$ almost surely. So a scalar system (2.1) is neither almost surely stable nor almost surely unstable when $\lambda_1^{p_1}\lambda_2^{p_2} \ldots \lambda_N^{p_N} = 1$. Second, note that the $\lambda_1, \ldots, \lambda_N$ are closely related to the matrix norm induced by the vector 2-norm in $\mathcal{R}^n$. This suggest the following more general sufficient condition for a.s. stability.

Theorem 2.2: *If there exists a matrix norm* $\|\cdot\|$, *such that*

$$\|H(1)\|^{p_1}\|H(2)\|^{p_2} \ldots \|H(N)\|^{p_N} < 1$$

then the system (2.1) *is almost surely stable.*

Proof: Since

$$\|H(\sigma_k) \ldots H(\sigma_1)\| \leqslant \|H(\sigma_k)\| \ldots \|H(\sigma_1)\|$$

following arguments similar to those used in the proof of (1) of Theorem 2.1, we obtain the desired result. □

Next, we comment on the importance of this result. From Desoer and Vidyasagar (1975), we know that the spectral radius $\rho(A)$ of a matrix A is given by

$$\rho(A) = \inf_{|\cdot| \in \mathcal{N}} [\sup_{x \in C^n} (|Ax|/|x|)]$$

where $\mathcal{N}$ is the set of all vector norms on C^n. This means that for any $\varepsilon > 0$, there exists an induced matrix norm $\|\cdot\|$ such that $\|A\| \leqslant \rho(A) + \varepsilon$. It follows that A is a stable matrix if and only if there exists an induced matrix norm so that $\|A\| < 1$. Proper choice of a vector norm can improve the stability region determined by Theorem 2.2. One possible consideration is as follows. For any non-singular matrix P, the vector norm $|x| = |Px|_2$ induces a matrix norm given by $\|A\| = \|PAP^{-1}\|_2$, where 2 denotes the 2-norm, i.e. the euclidean norm. Thus, we can try to solve the following optimization problem

$$\min_P \|PH(1)P^{-1}\|_2^{p_1}\|PH(2)P^{-1}\|_2^{p_2} \ldots \|PH(N)P^{-1}\|_2^{p_N}$$

If there is an optimal solution P^* and the optimal value of the objective function is less than unity, then (2.1), is almost surely stable. On the other hand, for a given P

$$D_P \stackrel{\text{def}}{=}$$

$$\{(H(1), \ldots, H(N)): \|PH(1)P^{-1}\|_2^{p_1}\|PH(2)P^{-1}\|_2^{p_2} \ldots \|PH(N)P^{-1}\|_2^{p_N} < 1\}$$

defines a stability region in the parameter space

$$\overbrace{\mathcal{R}^{n\times n} \times \mathcal{R}^{n\times n} \times \ldots \times \mathcal{R}^{n\times n}}^{N \text{ times}}$$

One may try to maximize the volume of D_P by a proper choice of P. This is a common procedure in applying Lyapunov's second method, i.e. try to select a suitable Lyapunov function from a given family so as to maximize the associated stability domain. A systematic approach to the optimization problem posed here for a more general family of systems is currently under investigation and will be presented at a later time. The following example demonstrates the application of the above results.

Example 2.1: Let

$$H(1) = \begin{bmatrix} 0{\cdot}9 & 1 \\ 0 & 0{\cdot}9 \end{bmatrix}, \quad H(2) = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad p_1 = p_2 = 0{\cdot}5$$

As $\lambda_1 = \lambda_{\max}(H(1)'H(1)) = 2{\cdot}3546$, $\lambda_2 = \lambda_{\max}(H(2)'H(2)) = 1$ and $\lambda_1^{p_1}\lambda_2^{p_2} = 1{\cdot}5345 > 1$, we can not use Theorem 2.1 directly.

Let $P = \begin{bmatrix} 1 & 0 \\ 0 & 100 \end{bmatrix}$, then we obtain

$$\|PH(1)P^{-1}\|^{p_1}\|PH(2)P^{-1}\|^{p_2} = \left\| \begin{bmatrix} 0{\cdot}9 & 0{\cdot}01 \\ 0 & 0{\cdot}9 \end{bmatrix} \right\|^{0{\cdot}5} \left\| \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\|^{0{\cdot}5} = 0{\cdot}951 < 1$$

Using Theorem 2.2, we conclude that the system (2.1) is a.s. stable. □

While almost sure (sample path) stability is the most desired stability property for engineering applications, moment stability concept are also important. In particular, the second ($\delta = 2$) moment stability is directly associated with solutions of the linear quadratic optimal control problem for jump linear systems. Considerable research has been devoted to the moment stability problem. Kozin's survey (Kozin 1969) clarified some confusion regarding the concepts of almost sure (sample path) and moment stability. It is well-known that second moment stability implies almost sure stability and, in many cases, second moment stability criteria are too conservative to be of practical value. The Lyapunov exponent approach to a.s. stability is a promising way of determining necessary and sufficient conditions for a.s. stability. Unfortunately, it is computationally difficult to evaluate the exponents and obtain numerical criteria for stability. Recently, more research effort has been devoted to revealing the relationship between δ-moment and almost-sure stability (Arnold 1982, Feng *et al*. 1992). It is expected that for jump linear systems, the stability region for δ-moment stability will monotonically increase to the almost-sure stability region. Therefore, a sufficient condition for δ-moment stability for δ small may yield a reasonable estimate of the a.s. stability region. Next, we present some of basic results in this direction which we have obtained so far.

An i.i.d. sequence is a special case of a Markov chain, so we can use the result in (Ji *et al*. 1991) to obtain a second moment stability criteria as follows.

Theorem 2.3: *A necessary and sufficient condition for the system* (2.1) *to be second moment* ($\delta = 2$) *stable is that for any N positive definite matricies* Q_1, Q_2, ..., Q_N, *there exists N positive definite matrices* P_1, P_2, ..., P_N *so that the following equations hold*

$$H'(i)\left(\sum_{j=1}^{N} p_j P_j\right)H(i) - P_i = -Q_i \quad i \in \underline{N}$$

The above condition involves the simultaneous solution of N coupled matrix equations. In order to use the theorem, we need to solve this set of matrix equations, which may be too complicated. For the i.i.d. case, we can reduce the above N matrix equations to one matrix equation. We obtain the following new results.

Theorem 2.4: *A necessary and sufficient condition for the system* (2.1) *to be second moment stable is that for any positive definite matrix Q, there exists a* *positive definite matrix R so that the following equation holds*

$$\sum_{j=1}^{N} p_j H(j)^{\mathrm{T}} R H(j) - R = -Q$$

Proof: It is sufficient to show that the two sets of equations in Theorems 2.3 and 2.4 are equivalent. Without loss of generality, we can assume that $Q_1 = Q_2 = \cdots = Q_N = Q = I$. Suppose that there exists positive definite matrices $P_1, P_2, \ldots, P_N$ so that

$$H'(j)\left(\sum_{i=1}^{N} p_i P_i\right) H(j) - P_j = -I$$

Multiplying both sides by p_j and summing from $j = 1$ to $j = N$, we obtain

$$\sum_{j=1}^{N} p_j H'(j)\left(\sum_{i=1}^{N} p_i P_i\right) H(j) - \sum_{j=1}^{N} p_j P_j = -I$$

Let $R = \sum_{i=1}^{N} p_i P_i$, which is also a positive definite matrix. Then R is a solution of the matrix equation in Theorem 2.4 for $Q = I$.

Conversely, suppose that R is the solution of the matrix equation in Theorem 2.4, let $P_i = I + H'(i)RH(i)$, then P_i $(1 \leqslant i \leqslant N)$ are positive definite matrices satisfying

$$\begin{aligned} H'(i)\left(\sum_{j=1}^{N} p_j P_j\right) H(i) - P_i &= H'(i)\left(\sum_{j=1}^{N} p_j (I + H'(i)RH(i))\right) H(i) - P_i \\ &= H'(i)\left(I + \sum_{j=1}^{N} p_j H'(j)RH(j)\right) H(i) - P_i \\ &= H'(i)RH(i) - P_i = -I \end{aligned}$$

This implies that $P_1, P_2, \ldots, P_N$ is a solution of the matrix equation in Theorem 2.3. This completes the proof. □

Although the above theorems provide a necessary and sufficient condition for second moment stability, we have to solve the above matrix equations. We now present some simpler sufficient conditions for second moment stability.

Theorem 2.5: *The system* (2.1) *is second moment stable, if one of the following conditions holds.*

(1) $H \overset{\text{def}}{=} E\{H'(\sigma_0)H(\sigma_0)\} = \sum_{j=1}^{N} p_j H'(j)H(j)$ *is a stable matrix, i.e.* $\rho(H) < 1$.

(2) $p_1\lambda_1 + p_2\lambda_2 + \cdots + p_N\lambda_N < 1$ *with* $\lambda_i = \lambda_{\max}(H(i)'H(i)) = \rho(H(i)'H(i))$ *as defined before.*

(3) $H(i)'H(i)$ *is stable matrix, i.e.* $\rho(H'(i)H(i)) < 1$, $i = 1, 2, \ldots, N$.

Proof: Let $\sigma = \rho(H)$. Then, for any $x \in \mathcal{R}^n$, we have $x'Hx \leqslant \sigma x'x$ and it follows that

$$\begin{aligned} Ex_k'x_k &= E\{x_{k-1}'H'(\sigma_{k-1})H(\sigma_{k-1})x_{k-1}\} \\ &= E\{E\{x_{k-1}'H'(\sigma_{k-1})H(\sigma_{k-1})x_{k-1}|x_{k-1}\}\} \\ &= E\{x_{k-1}'Hx_{k-1}\} \leqslant \sigma E\{x_{k-1}'x_{k-1}\} \leqslant \cdots \leqslant \sigma^k E\{x_0'x_0\} \end{aligned}$$

Thus, $\rho(H) < 1$ implies that (2.1) is (exponentially) second moment stable. This proves that (1) is a sufficient condition for second moment stability. Because $H(i)'H(i)$ and H are positive semi-definite matrices, $\rho(H) = \|H\|$ and $\lambda_i = \|H(i)'H(i)\|$. Then

$$\rho(H) = \|H\| = \left\|\sum_{i=1}^{N} p_i H(i)'H(i)\right\| \leq \sum_{i=1}^{N} p_i \|H(i)'H(i)\| = \sum_{i=1}^{N} p_i \lambda_i$$

It is clear that condition (2) implies (1). This shows the sufficiency of (2). Similarly, condition (3) clearly implies (2) and thus, (3) is also sufficient. □

Before dealing with the general δ-moment stability problem, we briefly examine the concept of mean stability. The following simple result illustrates the fact that mean stability of a jump linear system is closely related to the concept of S stability of an interval dynamic system.

Theorem 2.6: *The jump linear system* (2.1) *is mean stable if and only if the matrix* $\sum_{i=1}^{N} p_i H(i)$ *is stable.*

Proof: By independence of the sequence of $\{\sigma_k\}$, we have

$$\begin{aligned} E\{x_k\} &= E\{H(\sigma_{k-1})x_{k-1}\} = E\{H(\sigma_{k-1})\}E\{x_{k-1}\} = E\{H(\sigma_0)\}E\{x_{k-1}\} \\ &= \cdots = \left(\sum_{i=1}^{N} p_i H(i)\right)^k E\{x_0\} \end{aligned}$$

The result follows directly. □

The mean stability criteria given above is closely related to the so-called S stability for interval dynamic systems. Define

$$S(A_1, A_2, \ldots, A_N) = \left\{\sum_{i=1}^{N} p_i A_i \middle| p_i \geq 0, \sum_{i=1}^{N} p_i = 1\right\}$$

We say that S is stable if each matrix in S is stable. Then the problem is: under what conditions, is S stable? In general, the convex hull S is not stable even though the vertices $A_1, A_2, \ldots, A_N$ are stable. For example

$$A_1 = \begin{bmatrix} 0{\cdot}5 & 10 \\ 0 & 0{\cdot}5 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0{\cdot}5 & 0 \\ 10 & 0{\cdot}5 \end{bmatrix}$$

Even though A_1, A_2 are stable, it is easy to show that $0{\cdot}5A_1 + 0{\cdot}5A_2$ is not stable. For this problem, we have the following simple testable sufficient conditions for S stability. The proof is omitted here and is available from the authors (Fang *et al*. 1992 a, b).

Theorem 2.7: *The following statements hold.*

(*a*) *If $A_1, A_2, \ldots, A_N$ are stable symmetric matrices, then $S(A_1, A_2, \ldots, A_N)$ is stable.*

(*b*) *If $A_1, A_2, \ldots, A_N$ can be simultaneously transformed into upper (or lower) triangular forms by a similarity transformation, then S is stable if and only if $A_1, A_2, \ldots, A_N$ are stable.*

(*c*) *If $A_1, A_2, \ldots, A_N$ pairwise commute and are stable, then S is stable.*

(*d*) *If there exists a non-singular matrix P so that the singular values of PA_iP^{-1} are less than unity for all $i \in \underline{N}$, then S is stable.*

(*e*) *If for each $i \in \underline{N}$, A_i can be diagonalized by the same unitary transformation, and each A_i is stable, then S is stable.*

(*f*) *If there exists a positive definite matrix P such that $A_i'PA_i - P$ is negative definite for each $i \in \underline{N}$, then S is stable.*

S stability is closely related to the robustness property of interval dynamical systems. More general results are still under investigation and will be reported on at a later time. It is obvious that if $H(i) = A_i$ ($i = 1, 2, \ldots, N$) satisfies one of the conditions in Theorem 2.7, then the system (2.1) is mean stable for any probability distribution $\{p_1, p_2, \ldots, p_N\}$. This can be interpreted as a robustness property of mean stability against randomness in the system structure! We conclude this section with an example.

Example 2.11: Let

$$H(1) = \begin{bmatrix} 2\cdot 2 & -3\cdot 4 \\ 0\cdot 7 & -0\cdot 9 \end{bmatrix}, \quad H(2) = \begin{bmatrix} 15\cdot 3 & -30\cdot 4 \\ 7\cdot 2 & -14\cdot 3 \end{bmatrix}$$

Using the similarity transformation

$$T = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}$$

then we have

$$TH(1)T^{-1} = \begin{bmatrix} 0\cdot 5 & 1 \\ 0 & 0\cdot 8 \end{bmatrix}, \quad TH(2)T^{-1} = \begin{bmatrix} 0\cdot 1 & 8 \\ 0 & 0\cdot 9 \end{bmatrix}$$

Using (*b*) of Theorem 2.7, $p_1H(1) + p_2H(2)$ is always stable for any probability distribution $\{p_1, p_2\}$. From Theorem 2.6, we conclude that the system (2.1) is always mean stable for any i.i.d. random form process $\{\sigma_k\}$. □

3. δ-moment stability and its relationship to almost sure stability

In § 2, we have studied the almost-sure, second-moment and mean-stability properties of the jump linear system (2.1). As mentioned previously, the δ-moment stability criteria and the relationship between almost-sure and δ-moment stability is important because it can serve as the basis for developing sufficient conditions for almost-sure stability that may not be as conservative as the second moment stability criteria. In this section, we present δ-moment stability criteria and discuss the relationship between almost sure and δ-moment stability for the jump linear system (2.1).

Theorem 3.1: *The system* (2.1) *is δ-moment stable, if one of the following conditions holds.*

(*a*) $p_1\lambda_1^{\delta/2} + p_2\lambda_2^{\delta/2} + \cdots + p_N\lambda_N^{\delta/2} < 1$. *Where* $\lambda_i = \lambda_{\max}(H(i)'H(i))$ *as defined previously.*

(*b*) *If there exists an induced matrix norm $\|\cdot\|$ such that*

$$p_1\|H(1)\|^\delta + p_2\|H(2)\|^\delta + \cdots + p_N\|H(N)\|^\delta < 1$$

in particular, for a scalar system ($n = 1$), (a) and/or (b) is necessary and sufficient for δ-moment stability.

Proof: For any positive semi-definite matrix A, we have $(x'Ax)^\delta \leqslant (\lambda_{\max}(A))^\delta (x'x)^\delta$ for all $x \in \mathfrak{R}^n$. Then, it follows that

$$\begin{aligned} E\|x_k\|^\delta &= E\{x'_{k-1}H'(\sigma_{k-1})H(\sigma_{k-1})x_{k-1}\}^{\delta/2} \\ &\leqslant E\{\lambda_{\max}^{\delta/2}(H'(\sigma_{k-1})H(\sigma_{k-1}))(x'_{k-1}x_{k-1})^{\delta/2}\} \\ &= E\{\lambda_{\max}^{\delta/2}(H'(\sigma_{k-1})H(\sigma_{k-1})\}E\|x_{k-1}\|^\delta \\ &= E\{\lambda_{\max}^{\delta/2}(H'(\sigma_0)H(\sigma_0))\}E\|x_{k-1}\|^\delta \\ &\leqslant \cdots \leqslant (E\lambda_{\max}^{\delta/2}(H'(\sigma_0)H(\sigma_0)))^k E\|x_0\|^\delta \end{aligned}$$

By definition

$$E\lambda_{\max}^{\delta/2}(H'(\sigma_0)H(\sigma_0)) = p_1\lambda_1^{\delta/2} + p_2\lambda_2^{\delta/2} + \cdots + p_N\lambda_N^{\delta/2}$$

Which yields

$$E\|x_k\|^\delta \leqslant (p_1\lambda_1^{\delta/2} + \cdots + p_N\lambda_N^{\delta/2})^k E\|x_0\|^\delta$$

Therefore, if (a) holds, then the system (2.1) is δ-moment stable. Similarly, for the sufficiency of (b), we have

$$\begin{aligned} E\|x_k\|^\delta &= E\{\|H(\sigma_{k-1}) \ldots H(\sigma_0)x_0\|^\delta\} \\ &\leqslant E\{\|H(\sigma_{k-1})\|^\delta\} \ldots E\{\|H(\sigma_0)\|^\delta\}E\{\|x_0\|^\delta\} \\ &= (E\{\|H(\sigma_0)\|^\delta\})^k E\{\|x_0\|\} \\ &= (p_1\|H(1)\|^\delta + \cdots + p_N\|H(N)\|^\delta)^k E\{\|x_0\|^\delta\} \end{aligned}$$

The result follows directly. Finally, we observe that for a scalar system, the conditions (a) and (b) are the same. Hence, we need only to show that (a) is also a necessary condition for δ-moment stability in this case. However, for a scalar system, $H(i)$ for $i \in \underline{N}$ are scalars and we obtain

$$\begin{aligned} E\|x_k\|^\delta &= E\{\|H(\sigma_{k-1})\|^\delta \ldots \|H(\sigma_0)\|^\delta \|x_0\|\} \\ &= (E\|H(\sigma_0)\|^\delta)^k E\|x_0\|^\delta \\ &= (p_1\lambda_1^{\delta/2} + \cdots + \lambda_N^{\delta/2})^k E\|x_0\|^\delta \end{aligned}$$

Hence, $E\|x_k\|^\delta \to 0$ if and only if $p_1\lambda_1^{\delta/2} + p_2\lambda_2^{\delta/2} + \cdots + \lambda_N^{\delta/2} < 1$. This completes the proof. □

As we mentioned at the beginning of the paper, all the results in this work can be extended to treat the case when $\{\sigma_k\}$ is a Markov chain. Next, we present a theorem for the markovian case, which is the corresponding generalization of Theorem 3.1 proved above.

Theorem 3.2: *Suppose that the form process $\{\sigma_k\}$ in (2.1) is a finite-state Markov chain with the probability transition matrix P. Let $D = \mathrm{diag}\{\|H(1)\|^\delta, \|H(2)\|^\delta, \ldots, \|H(N)\|^\delta\}$. Then, a sufficient condition for (2.1) to be δ-moment stable is that DP is a stable matrix, i.e. $\rho(DP) < 1$. Furthermore, for a scalar system ($n = 1$), if the strong mixing condition $P = (p_{ij})_{N\times N} > 0$ is satisfied,* i.e. *$p_{ij} > 0$ for all i, $j \in \underline{N}$, then, $\rho(DP) < 1$ is also a necessary condition for δ-moment stability.* □

Proof: Let $a_i = \|H(i)\|^\delta$. Then

$$\begin{aligned} E\|H(\sigma_k)\ \dots\ H(\sigma_0)\|^\delta &\leq E\|H(\sigma_k)\|^\delta \dots \|H(\sigma_0)\|^\delta \\ &= \sum_{i_0,i_1,\dots,i_k} p_{i_0}p_{i_0i_1}\dots p_{i_{k-1}i_k}\|H(i_k)\|^\delta \dots \|H(i_0)\|^\delta \\ &= \sum_{i_0,i_1,\dots,i_k} p_{i_0}p_{i_0i_1}\dots p_{i_{k-1}i_k}a_{i_k}\dots a_{i_1}a_{i_0} \\ &= \sum_{i_0,i_1,\dots,i_k} (p_{i_0}a_{i_0})(p_{i_0i_1}a_{i_1})\dots(p_{i_{k-1}i_k}a_{i_k}) \\ &= (\pi_0 D)(PD)\dots(PD)(Pb) = \pi_0(DP)^k b \end{aligned}$$

where $b = (a_1, a_2, \dots, a_N)'$ and $\pi_0 = (p_1, p_2, \dots, p_N)$ is the initial probability distribution. If DP is stable, we have $\lim_{k\to\infty}(DP)^k = 0$. It follows that $\lim_{k\to\infty} E\|H(\sigma_k)\ \dots\ H(\sigma_0)\|^\delta = 0$. This implies that (2.1) is δ-moment stable. This proves the first statement. For the scalar case, without loss of generality, we assume that $a_i = \|H(i)\|^\delta \neq 0$. According to the strong mixing assumption, P is a positive matrix, and so is DP. Because PD and DP have the same eigenvalues, DP is stable if and only if PD is stable. Let $\lambda_{\max}$ be the largest eigenvalue of PD, which is, in fact, the spectral radius of PD, then from Perron's theorem (Horn and Johnson 1985), there exists a positive (row) eigenvector x for $\lambda_{\max}$, i.e. $x'PD = \lambda_{\max}x'$. As (2.1) is δ-moment stable for any initial distribution, choose initial distributions which are the rows of P, we then obtain

$$\lim_{k\to\infty} P(DP)^k b = \lim_{k\to+\infty}(PD)^{k+1}D^{-1}b = \lim_{k\to\infty}(PD)^{k+1}e = 0$$

where $e = (1, 1, \dots, 1)'$. Therefore, for

$$x = (x_1, x_2, \dots, x_N)', \ \lim_{k\to\infty} x'(PD)^{k+1}e = 0$$

It follows that

$$\lim_{k\to+\infty} x'(PD)^{k+1}e = \lim_{k\to+\infty}\lambda_{\max}^{k+1}x'e = \lim_{k\to+\infty}\lambda_{\max}^{k+1}(x_1 + \cdots + x_N) = 0$$

Because $x_1 + \cdots + x_N > 0$, this implies that $\lim_{k\to\infty}\lambda_{\max}^{k+1} = 0$ and $\lambda_{\max} < 1$. That is, DP is a stable matrix. This completes the proof. □

We know that an i.i.d. sequence is a special type of Markov chain. To be more specific, if $\{\sigma_k\}_{k=0}^{+\infty}$ is an i.i.d. sequence with common distribution $p = (p_1, \dots, p_N)$ and $\{\xi_j\}_{j=1}^{+\infty}$ is a Markov chain with a transition matrix $P = (p_{ij})_{N\times N}$ and initial distribution π_0, then $\{\sigma_k\}_{k=0}^{+\infty}$ and $\{\xi_j\}_{j=1}^{+\infty}$ have the same set of finite dimensional distributions as long as $p_{ij} = p_j$ for all $i, j \in \underline{N}$, regardless of π_0. Therefore, one can identify the two sequences $\{\sigma_k\}_{k=1}^{+\infty}$ and $\{\pi_k\}_{k=1}^{+\infty}$ with each other. From this, it is expected that we can recover the δ-moment stability criteria for the i.i.d. case from the above result. We justify this claim next.

Lemma 3.1: *Let A be a non-negative matrix, i.e. a matrix with non-negative entries. Then, for any positive vector x where $x = (x_1, \dots, x_n)'$, we have*

$$\min_{1\leq i\leq n}\frac{1}{x_i}\sum_{j=1}^n a_{ij}x_j \leq \rho(A) \leq \max_{1\leq i\leq n}\frac{1}{x_i}\sum_{j=1}^n a_{ij}x_j \tag{3.1}$$

Furthermore, if there exists a positive vector x so that $Ax = \lambda x$ for a positive λ, then $\rho(A) = \lambda$.

Proof: Equation (3.1) can be found in Horn and Johnson (1985). Assume that $Ax = \lambda x$ for some x and λ positive. Then, $\sum_{j=1}^{n} a_{ij} x_j = \lambda x_i$. By applying (3.1), we obtain $\rho(A) = \lambda$. □

Theorem 3.3: *Let $P = (p_{ij})_{N\times N}$ be a transition matrix satisfying $p_{ij} = p_j > 0$ for all $i, j \in \underline{N}$. Then, DP is stable if and only if*

$$p_1 \|H(1)\|^\delta + \cdots + p_N \|H(N)\|^\delta < 1 \tag{3.2}$$

i.e. for an i.i.d. form process with common distribution $(p_1, \ldots, p_N)$, a sufficient condition for δ-moment stability is (3.2).

Proof: Let $A = (DP)'$. Then, A is non-negative and the following equation holds

$$A\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_N \end{bmatrix} = \begin{bmatrix} a_1 p_1 & a_2 p_1 & \cdots & a_N p_1 \\ a_1 p_2 & a_2 p_2 & \cdots & a_N p_2 \\ \vdots & \vdots & & \vdots \\ a_1 p_N & a_2 p_N & \cdots & a_N p_N \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_N \end{bmatrix}$$

$$= \begin{bmatrix} (p_1 a_1 + \cdots + p_N a_N) p_1 \\ (p_1 a_1 + \cdots + p_N a_N) p_2 \\ \vdots \\ (p_1 a_1 + \cdots + p_N a_N) p_N \end{bmatrix} = (p_1 a_1 + \cdots + p_N a_N)\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_N \end{bmatrix}$$

From Lemma 3.1, $p_1 a_1 + \cdots + p_N a_N$ is the spectral radius of $A = DP$. Thus DP is stable if and only if

$$p_1 a_1 + \cdots + p_N a_N = p_1 \|H(1)\|^\delta + p_2 \|H(2)\|^\delta + \cdots + p_N \|H(N)\|^\delta < 1$$

□

We know that for an ergodic markovian form process, the almost sure stability property is independent of the initial distribution. From the above results, it is very tempting to conjecture that this may also be true for δ-moment stability. In particular, for a scalar system with an ergodic markovian form process, one may conjecture that the δ-moment stability condition is $\pi_1 a_1 + \cdots + \pi_N a_N < 1$ with $\pi = (\pi_1, \ldots, \pi_N)$ being the unique invariant distribution (a_i is defined as above). Equivalently, the stability of DP may be equivalent to $\pi_1 a_1 + \cdots + \pi_N a_N < 1$. Unfortunately, this is not true as illustrated by the following example.

Example 3.1: Let $H(1) = 1{\cdot}9$ and $H(2) = 0{\cdot}5$, and

$$P = \begin{bmatrix} 0{\cdot}1 & 0{\cdot}9 \\ 0{\cdot}8 & 0{\cdot}2 \end{bmatrix}$$

For the first moment ($\delta = 1$), it is easy to see that the eigenvalues of DP are $0{\cdot}973$ and $-0{\cdot}683$. Hence, DP is stable and (2.1) is first-moment stable from Theorem 3.2. However, the unique invariant measure is (8/17, 9/17) and for $\delta = 1$, $\pi_1 a_1 + \pi_2 a_2 = (8/17) \times 1{\cdot}9 + (9/17) \times 0{\cdot}5 = 19{\cdot}7/17 > 1$. □

The above criteria for δ-moment stability are easily testable. In the remainder of this section, we study the relationship between δ-moment stability and almost sure stability. We show that for a scalar jump linear system, the stability regions Σ^δ in the space of system parameters associated with δ-moment stability are monotonically increasing as δ decreases to 0 and Σ^δ tends to the almost-sure stability region Σ^a monotonically from the interior of Σ^a. Similar results have been obtained for a scalar continuous-time jump linear system with a markovian form process by Feng *et al*. (1992) and Feng (1990). We first prove a lemma which plays a central role in the development.

Lemma 3.2: *For $\lambda_i \geq 0$, $i \in \underline{N}$ fixed, define a function $f: \mathfrak{R} \to \mathfrak{R}$ by*

$$f(x) = \begin{cases} (p_1\lambda_1^x + p_2\lambda_2^x + \cdots + p_N\lambda_N^x)^{1/x} & \text{if } x \neq 0 \\ \lambda_1^{p_1}\lambda_2^{p_2} \ldots \lambda_N^{p_N} & \text{if } x = 0 \end{cases}$$

Then, $f(x)$ is continuous and non-decreasing on $[0, \infty)$.

Proof: It is easy to see that $f(x)$ is continuous on $\mathfrak{R}\backslash\{0\}$. Hence, it suffices to show that it is continuous at $x = 0$. In fact, using L'Hopital's rule, we have

$$\begin{aligned}\lim_{x\to 0}(p_1\lambda_1^x + \cdots + p_N\lambda_N^x)^{1/x} &= \exp\left(\lim_{x\to 0}\frac{\log(p_1\lambda_1^x + \cdots + p_N\lambda_N^x)}{x}\right) \\ &= \exp\left(\lim_{x\to 0}\frac{p_1\lambda_1^x\log\lambda_1 + \cdots + p_N\lambda_N^x\log\lambda_N}{p_1\lambda_1^x + \cdots + p_N\lambda_N^x}\right) \\ &= \lambda_1^{p_1}\lambda_2^{p_2} \ldots \lambda_N^{p_N}\end{aligned}$$

Thus, $f(x)$ is continuous at 0. Next, we show that f is non-decreasing on $[0, +\infty)$. For any $x_1, x_2 \in [0, +\infty)$ with $x_1 < x_2$, let $\beta = x_1/x_2 < 1$ and $g(x) \overset{\text{def}}{=} x^\beta$. As $\beta < 1$, $g(x)$ is concave. From Jensen's inequality, we obtain

$$\sum_{i=1}^N p_i g(\lambda_i^{x_2}) \leq g\left(\sum_{i=1}^N p_i\lambda_i^{x_2}\right)$$

From this, we have $f(x_1) \leq f(x_2)$. Thus, $f(x)$ is non-decreasing on $[0, +\infty)$. This completes the proof.

Now, define

$$\Sigma^a = \{(\lambda_1, \lambda_2, \ldots, \lambda_N) | \lambda_1^{p_1}\lambda_2^{p_2} \ldots \lambda_N^{p_N} < 1, \lambda_j \geq 0\}$$
$$\Sigma^\delta = \{(\lambda_1, \lambda_2, \ldots, \lambda_N) | p_1\lambda_1 + p_2\lambda_2 + \cdots + p_N\lambda_N < 1, \lambda_j \geq 0\}$$

By the above results, we see that Σ^a is the almost sure stability region and Σ^δ is the δ-moment stability region in the space of parameters of $(H(1), \ldots, H(N))' \in \mathfrak{R}^N$. We have the following important theorem.

Theorem 3.4: *For the scalar jump linear system* (2.1)

(*a*) *for any $\delta_1, \delta_2 > 0$ with $\delta_1 < \delta_2$, we have $\Sigma^{\delta_2} \subset \Sigma^{\delta_1} \subset \Sigma^a$.*

(*b*) $\lim_{\delta\to 0}\Sigma^\delta = \bigcup_{\delta>0}\Sigma^\delta = \Sigma^a$.

Proof: The proof is a direct application of Lemma 3.2. □

Remark: This theorem is very important, because it reveals the relationship between the almost-sure stability and the δ-moment stability. First of all, it says that δ-moment stability implies almost-sure stability for any $\delta > 0$. Secondly, as we mentioned previously, almost-sure stability is often a desired stability property; and a testable necessary and sufficient condition for almost-sure stability, however, is extremely difficult to obtain in general (the Lyapunov exponent approach seems promising). Hence, if we can use the criteria in this section to test δ-moment stability, in particular for small $\delta > 0$, then we can obtain a sufficient condition for almost-sure stability. This theorem says that this sufficient condition for almost-sure stability derived from δ-moment stability is 'close' to being necessary for $\delta > 0$ 'small'. In this sense, almost-sure stability can be studied using criteria for δ-moment stability. □

We conclude this section with two examples. These examples illustrate that for a certain class of systems, the criteria for almost-sure stability obtained in this paper are better than those obtained from second moment stability.

Example 3.2: Let $H(1) = 2{\cdot}5$, $H(2) = 0{\cdot}1$, and $p_1 = p_2 = 0{\cdot}5$, then $\lambda_1 = 6{\cdot}25$ and $\lambda_2 = 0{\cdot}1$. Thus

$$\lambda_1^{p_1}\lambda_2^{p_2} = \sqrt{6{\cdot}25 \times 0{\cdot}01} = 0{\cdot}25 < 1$$

$$p_1\lambda_1 + p_2\lambda_2 = \frac{6{\cdot}25 + 0{\cdot}01}{2} = 3{\cdot}13 > 1$$

and from Theorem 2.1 and Theorem 2.5, we conclude that the system is almost surely stable, but not second-moment stable. □

Example 3.3: Let $\{\sigma_k\}$ be a two-state i.i.d. sequence, and let $H(1) = \alpha$, and $H(2) = \beta$, and $p_1 = p_2 = 0{\cdot}5$. Then, according to Theorem 3.1, the system (2.1) is δ-moment stable if and only if

$$\frac{|\alpha|^\delta + |\beta|^\delta}{2} < 1$$

and it is almost surely stable if and only if

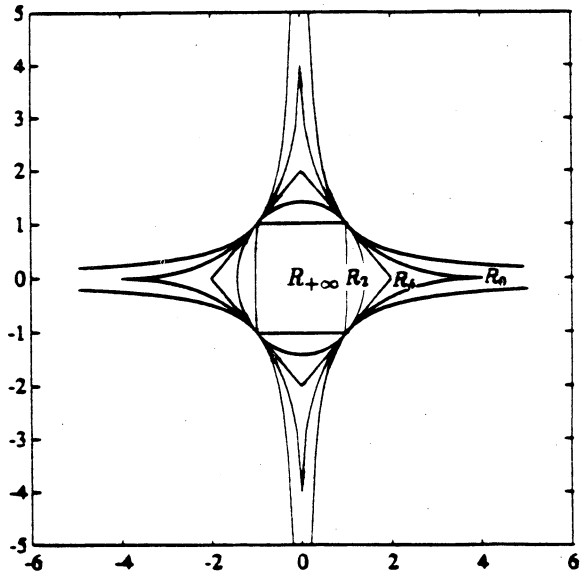
$$|\alpha|\,|\beta| < 1$$ □

The Figure illustrates the stability regions.

In Fig. 1, R_0 denotes the almost-sure stability region, which is the open connected region enclosed by the four (disconnected) hyperbolic curves. R_δ is the δ-moment stability region with a $\delta < 1$, which is the open connected bounded region enclosed by the next four (connected) hyperbolic curves. The diamond-shaped region denoted by R_1 is the first-moment stability region. The open connected region R_2 enclosed by the ellipse is the second-moment stability region. Finally, the open connected square $R_{+\infty}$ is the δ-moment stability region for $\delta = +\infty$. Indeed, we have $R_\infty \subset R_2 \subset R_1 \subset R_0$, and as δ decreases to 0^+, R_δ tends to R_0 monotonically. This is consistent with our previous analysis. A generalization of the above one-dimensional results to the general class of jump linear systems is currently under investigation and we will report on these results in a subsequent paper.

4. Systems with special commuting structures

In the previous sections, we have studied almost-sure and δ-moment stability of a jump linear system of the form (2.1). Simple testable sufficient conditions for stability are given. These conditions turn out to be also necessary for



Almost-sure and δ-moment stability regions.

one-dimensional systems. This suggests that the commutivity of the mode matrices $H(1), \ldots, H(N)$ is critical for necessity. Indeed, it is true that the effectiveness of these conditions depends on the eigen-structures of the mode matrices, in particular, the pairwise commutivity of $H(1), \ldots, H(N)$. In this section, we attempt to examine commuting structures in more detail and some testable necessary and sufficient conditions for almost sure stability are obtained. Although we only deal with systems with an i.i.d. form process, our results are still valid for the general Markovian case. The following theorem relates the almost sure stability to the spectral radii of $H(1), \ldots, H(N)$ when the matrices pairwise commute.

Theorem 4.1: *If $H(1), H(2), \ldots, H(N)$ pairwise commute, then*

$$\mu_1^{p_1}\mu_2^{p_2} \ldots \mu_N^{p_N} < 1$$

is a sufficient condition for (2.1) *to be almost surely stable, where $\mu_i = \rho(H(i))$ is the spectral radius of $H(i)$ for each $i \in \underline{N}$.*

To prove the theorem, we need the following lemma whose proof appears in the appendix.

Lemma 4.1: *Let $A \in \mathfrak{R}^{n\times n}$. Then, for $\varepsilon > 0$ and $\|\cdot\|$ a norm on $\mathfrak{R}^{n\times n}$ arbitrarily given, there exists a constant $M > 0$ such that*

$$\|A^k\| \leqslant M\alpha^k \quad \forall k > 0$$

Where $\alpha = |\lambda_{\max}(A)| + \varepsilon$.

Proof of Theorem 4.1: As the matrices $H(1), \ldots, H(N)$ pairwise commute, we have

$$x_k = H(1)^{\sum_{i=0}^{k-1} I_1(\sigma_i)} H(2)^{\sum_{i=0}^{k-1} I_2(\sigma_i)} \ldots H(N)^{\sum_{i=0}^{k-1} I_N(\sigma_i)} x_0$$

Hence, from Lemma 4.1, for any $\varepsilon > 0$, $\exists M_1, M_2, \ldots, M_N$ such that $\|H(i)^k\| \leq M_i(\mu_i + \varepsilon)^k$ for $1 \leq i \leq N$ and $k > 0$. It follows that

$$\|x_k\| \leq M_1(\mu_1 + \varepsilon)^{\sum_{i=0}^{k-1} I_1(\sigma_i)} \ldots M_N(\mu_N + \varepsilon)^{\sum_{i=0}^{k-1} I_N(\sigma_i)} \|x_0\|$$
$$= M_1 \ldots M_N \left((\mu_1 + \varepsilon)^{\frac{1}{k}\sum_{i=0}^{k-1} I_1(\sigma_i)} \ldots (\mu_N + \varepsilon)^{\frac{1}{k}\sum_{i=0}^{k-1} I_N(\sigma_i)} \right)^k \|x_0\|$$

Since $\mu_1^{p_1}\mu_2^{p_2} \ldots \mu_N^{p_N} < 1$, there is an $\varepsilon > 0$ so that $(\mu_1 + \varepsilon)^{p_1}(\mu_2 + \varepsilon)^{p_2} \ldots (\mu_N + \varepsilon)^{p_N} < 1$. Then, from the Law of Large Numbers, we conclude that $\|x_k\| \to 0$ as $k \to \infty$. This completes the proof. □

It is easy to see that for the one-dimensional case, the condition in the above theorem is also necessary. The following corollary of Theorem 4.1 is immediate.

Corollary 4.1: *If $H(1), H(2), \ldots, H(N)$ pairwise commute and are all stable matrices, then the system* (2.1) *is almost surely stable.*

In general, the condition in Theorem 4.1 is not necessary. The following demonstrates this fact and is illustrative for the later development.

Example 4.1:

$$H(1) = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix}, \quad H(2) = \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix}$$

A direct computation yields

$$x_k = \begin{bmatrix} \delta_1^{\frac{1}{k}\sum_{i=0}^{k-1} I_1(\sigma_i)} \alpha_1^{\frac{1}{k}\sum_{i=0}^{k-1} I_2(\sigma_i)} & 0 \\ 0 & \delta_2^{\frac{1}{k}\sum_{i=0}^{k-1} I_1(\sigma_i)} \alpha_2^{\frac{1}{k}\sum_{i=0}^{k-1} I_2(\sigma_i)} \end{bmatrix}^k x_0 \triangleq A(k)^k x_0$$

By applying the Law of Large Numbers, we see that the system (2.1) is almost surely stable if and only if

$$\delta_1^{p_1}\alpha_1^{p_2} < 1$$
$$\delta_2^{p_1}\alpha_2^{p_2} < 1$$

If we choose $\delta_1 = 2$, $\delta_2 = 0{\cdot}1$, $\alpha_1 = 0{\cdot}1$, $\alpha_2 = 2$ and $p_1 = p_2 = 0{\cdot}5$, then the system (2.1) is almost surely stable, even though $\mu_1^{p_1}\mu_2^{p_2} = 2 > 1$. □

The above example suggests the following generalization to systems with a more general type of commuting structure.

Theorem 4.2: *Suppose that $H(1), H(2), \ldots, H(N)$ pairwise commute. Then*

(*a*) *if $H(1), \ldots, H(N)$ can be simultaneously diagonalized to the following form*

$$T^{-1}H(j)T = \begin{bmatrix} \rho_{1,j} & & & \\ & \rho_{2,j} & & \\ & & \ddots & \\ & & & \rho_{n,j} \end{bmatrix} \quad (1 \leq j \leq N)$$

then the system (2.1) *is almost surely stable if and only if*

$$|\rho_{i,1}^{p_1}\rho_{i,2}^{p_2} \ldots \rho_{i,N}^{p_N}| < 1 \;\forall\, 1 \leq i \leq n$$

(*b*) *if $H(1), \ldots, H(N)$ can be simultaneously transformed to Jordan form* *with the corresponding diagonal elements* $\rho_{1,j}, \rho_{2,j}, \ldots, \rho_{n,j}$ *for* $j \in \underline{N}$, *then a necessary and sufficient condition for the system* (2.1) *to be almost surely stable is*

$$|\rho_{i,1}^{p_1}\rho_{i,2}^{p_2} \ldots \rho_{i,N}^{p_N}| < 1 \;\forall\, i \in \{1, \ldots, n\}$$

Clearly, the proof of (a) is similar to the one-dimensional case. To show (b), we need the following matrix lemma.

Lemma 4.2:

(i) *Any two Jordan blocks with the same dimension commute. Any diagonal matrix commuting with a Jordan block will be a scalar multiple of the identity matrix.*

(ii) *Any two Jordan form matrices A and B which commute can be written as matrix diagonal forms, in which the number of the diagonal submatrices are equal, and the corresponding submatrices,* J_i *and* $\bar{J}_i$ *have the same dimension, i.e.*

$$A = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_r \end{bmatrix}, \quad B = \begin{bmatrix} \bar{J}_1 & & & \\ & \bar{J}_2 & & \\ & & \ddots & \\ & & & \bar{J}_r \end{bmatrix}$$

where $\dim \bar{J}_i = \dim J_i$ *for* $1 \leq i \leq r$, *and* J_i, $\bar{J}_i$ *are either diagonal or Jordan blocks. These canonical forms will be referred to as* $D-J$ *forms.*

(iii) *If* $A_1, A_2, \ldots, A_k$ *commute and can be simultaneously transformed to Jordan forms, then they can be simultaneously transformed into* $D-J$ *forms.*

Proof: For the proof see the Appendix. □

Proof of Theorem 4.2: (a) is trivial. We only show (b) here. The necessity is obvious by examining the diagonal elements of matrix products of the Jordan forms. To show (b), it is enough to prove sufficiency.

As the matrices $H(1), H(2), \ldots, H(N)$ pairwise commute and can be simultaneously transformed to Jordan forms, from (iii) of Lemma 4.2, there is a non-singular T such that for each $j \in \underline{N}$,

$$H(j) = T^{-1} \operatorname{diag}\{J_{1,j}, J_{2,j}, \ldots, J_{r,j}\} T$$

where $J_{l,j}$ for $1 \leq l \leq r$ are either Jordan blocks or diagonal matrices, and $\dim J_{l,1} = \dim J_{l,2} = \cdots = \dim J_{l,N}$ for all $1 \leq l \leq r$. Thus, we have

$$\begin{aligned} x_k &= H(1)^{\sum_{i=0}^{k-1} I_1(\sigma_i)} \ldots H(N)^{\sum_{i=0}^{k-1} I_N(\sigma_i)} x_0 \\ &= T^{-1} \begin{bmatrix} J_{1,1}^{\sum_{i=0}^{k-1} I_1(\sigma_i)} \ldots J_{1,N}^{\sum_{i=0}^{k-1} I_N(\sigma_i)} & & \\ & \ddots & \\ & & J_{r,1}^{\sum_{i=0}^{k-1} I_1(\sigma_i)} \ldots J_{r,N}^{\sum_{i=0}^{k-1} I_N(\sigma_i)} \end{bmatrix} T x_0 \end{aligned}$$

Let $q_m = \sum_{i=0}^{k-1} I_m(\sigma_i)$ for $1 \leq m \leq N$. For simplicity, assume that $N = 2$.

Let us examine the diagonal block $J_{i,1}^{q_1} J_{i,2}^{q_2}$ with the corresponding eigenvalues of $J_{i,1}$, $J_{i,2}$ being α_i, β_i respectively.

Case 1: if both $J_{i,1}$ and $J_{i,2}$ are diagonal, then from (a), $J_{i,1}^{q_1} J_{i,2}^{q_2}$ converges to zero almost surely.

Case 2: if $J_{i,1}$ is a Jordan block, with dimension m, and $J_{i,2}$ is diagonal, then from (i) of Lemma 4.2, we see that $J_{i,2}$ is, in fact, a scalar multiple of the identity matrix, i.e. $J_{i,2} = \beta_i I$, then we obtain

$$J_{i,1}^{q_1} J_{i,2}^{q_2} = \beta_i^{q_2} \begin{bmatrix} \alpha_i & 1 & & \\ & \alpha_i & \ddots & \\ & & \ddots & 1 \\ & & & \alpha_i \end{bmatrix}^{q_1}$$

$$= \begin{bmatrix} \alpha_i^{q_1}\beta_i^{q_2} & \binom{q_1}{1}\alpha_i^{q_1-1}\beta_i^{q_2} & \cdots & \binom{q_1}{m-1}\alpha_i^{q_1-m+1}\beta_i^{q_2} \\ & \alpha_i^{q-1}\beta_i^{q_2} & \ddots & \vdots \\ & & \ddots & \vdots \\ & & & \alpha_i^{q-1}\beta_i^{q_1} \end{bmatrix}$$

Hence, from the Law of Large Numbers, we obtain

$$\lim_{k\to\infty} \alpha_i^{q_1/k}\beta_i^{q_2/k} \overset{\text{a.s.}}{=} \alpha_i^{p_1}\beta_i^{p_2}$$

From the sufficiency condition, $\alpha_i^{p_1}\beta_i^{p_2} < 1$, thus for any finite l

$$\lim_{k\to\infty} \binom{q_1}{l} \alpha_i^{q_1-l}\beta_i^{q_2} \overset{\text{a.s.}}{=} 0$$

This implies that $J_{i,1}^{q_1} J_{i,2}^{q_2}$ converges to zero almost surely.

Case 3: if $J_{i,1}$ and $J_{i,2}$ are both Jordan blocks with dimension m, then we have

$$J_{i,1}^{q_1} J_{i,2}^{q_2} = \begin{bmatrix} \alpha_i & 1 & & \\ & \alpha_i & \ddots & \\ & & \ddots & 1 \\ & & & \alpha_i \end{bmatrix}^{q_1} \begin{bmatrix} \beta_i & 1 & & \\ & \beta_i & \ddots & \\ & & \ddots & 1 \\ & & & \beta_i \end{bmatrix}^{q_2}$$

$$= \begin{bmatrix} \alpha_i^{q_1}\beta_i^{q_2} & \sum_{l=0}^{l=1}\binom{q_1}{l}\binom{q_2}{1-l}\alpha_i^{q_1-l}\beta_i^{q_2-1+l} & \cdots & \sum_{l=0}^{m-1}\binom{q_1}{l}\binom{q_2}{m-1-l}\alpha_i^{q_1-l}\beta_i^{q_2-m+l+1} \\ & \alpha_i^{q_1}\beta_i^{q_2} & \cdots & \\ & & \ddots & \vdots \\ & & & \alpha_i^{q_1}\beta_i^{q_2} \end{bmatrix}$$

By hypothesis, we know that $\alpha_i^{p_1}\beta_i^{p_2} < 1$, and for any finite integer l, we obtain

$$\binom{q_2}{l} \alpha_i^{q_1}\beta_i^{q_2-l} + \binom{q_1}{1}\binom{q_2}{l-1} \alpha_i^{q_1-1}\beta_i^{q_2-l+1} + \cdots + \binom{q_1}{l} \alpha_i^{q_1-l}\beta_i^{q_2}$$
$$\overset{\text{a.s.}}{=} O(k^l)(\alpha_i^{q_1/k}\beta_i^{(q_2-l)/k})^k + \cdots + O(k^l)(\alpha_i^{(q_1-l)/k}\beta_i^{q_2/k})^k \overset{\text{a.s.}}{\to} 0$$

where $O(a_n)$ denotes a quantity such that $O(a_n)/a_n$ is bounded as $a_n \to 0$. In the above derivation, we have used the fact that $\lim_{k\to\infty} k^l a^k = 0$ for any finite integer l, where $|a| < 1$. In this way, we can conclude that $J_{i,1}^{q_1} J_{i,2}^{q_2}$ converges to zero almost surely.

Combining the above discussion, we obtain the proof of sufficiency for $N = 2$. The case when $N > 2$ is similar. This completes the proof. □

Remark: As we already know, if the matrices $H(1)$, $H(2)$, ..., $H(N)$ are diagonalizable, then they commute if and only if they can be simultaneously transformed to diagonal forms. But, for the simultaneous transformation to

Jordan form, this does not hold; that is, commutivity does not guarantee simultaneous transformation to Jordan form. A simple example is

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

□

More generally, one may ask for a necessary and sufficient condition for the case when $H(1), \ldots, H(N)$ pairwise commute without additional restrictions in the eigenstructure of the matrices. The following results partially answer this question.

Theorem 4.3: *Suppose that $H(1)$, $H(2)$, ..., $H(N)$ pairwise commute. Then there is a unitary matrix $U \in \mathscr{C}^{n\times n}$ (the linear vector space of $n \times n$ matrices with complex entries) such that for each $j \in \underline{N}$, $U^*H(j)U$ is an upper triangular matrix with the diagonal elements $\{\rho_{1,j}, \rho_{2,j}, \ldots, \rho_{n,j}\}$. A necessary condition for* (2.1) *to be almost surely stable is*

$$|\rho_{i,1}^{p_1}\rho_{i,2}^{p_2}\ldots\rho_{i,N}^{p_N}| < 1 \;\forall\, 1 \leqslant i \leqslant n$$

Proof: The existence of such a unitary matrix U is given by Horn and Johnson (1985, p. 81). To show the necessity, first use the unitary transformation to change the matrix product into a product of upper triangular matrices, then use the diagonal elements together with the Law of Large Numbers and the Law of the Iterated Logarithm, the proof is then complete. □

We conjecture that the above condition is also sufficient. However, we have not obtained a proof of the sufficiency. The following two-dimensional result supports the above conjecture.

Theorem 4.4: *For two-dimensional ($n = 2$) commuting matricies $H(1)$, $H(2)$, ..., $H(N)$, the necessary condition for almost sure stability given in Theorem 4.3 is also sufficient.*

To prove this, we need the following lemma.

Lemma 4.3:

(i)
$$\begin{bmatrix} \alpha & b \\ 0 & \alpha \end{bmatrix}^k = \begin{bmatrix} \alpha^k & kb\alpha^{k-1} \\ 0 & \alpha^k \end{bmatrix}$$

(ii)
$$\begin{bmatrix} \alpha & a \\ 0 & \alpha \end{bmatrix}$$

and

$$\begin{bmatrix} \beta & b \\ 0 & \beta \end{bmatrix}$$

commute. If $a \neq 0$, and

$$\begin{bmatrix} \alpha & a \\ 0 & \alpha \end{bmatrix}$$

commutes with

$$\begin{bmatrix} \beta_1 & b \\ 0 & \beta_2 \end{bmatrix}$$

then $\beta_1 = \beta_2$.

(iii) *Let*

$$A = \begin{bmatrix} \alpha_1 & a \\ 0 & \alpha_2 \end{bmatrix} \quad \textit{and } B = \begin{bmatrix} \beta_1 & b \\ 0 & \beta_2 \end{bmatrix}$$

where $a \neq 0$, $b \neq 0$, $\alpha_1 \neq \alpha_2$, $\beta_1 \neq \beta_2$. *If A and B commute, then they can be simultaneously diagonalized as either*

$$\begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix}, \begin{bmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{bmatrix} \textit{ or } \begin{bmatrix} \alpha_2 & 0 \\ 0 & \alpha_1 \end{bmatrix}, \begin{bmatrix} \beta_2 & 0 \\ 0 & \beta_1 \end{bmatrix}$$

there are no other choices.

Proof: (i) and (ii) follow from a direct computation. We show (iii) next: let T be the matrix so that $T^{-1}AT$, $T^{-1}BT$ are diagonal, and let

$$T = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}$$

Suppose that

$$T^{-1}AT = \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix}$$

Then

$$AT = T\begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} \alpha_1 t_{11} + at_{21} & t_{12}\alpha_1 + at_{22} \\ \alpha_2 t_{21} & \alpha_2 t_{22} \end{bmatrix} = \begin{bmatrix} \alpha_1 t_{11} & \alpha_2 t_{12} \\ \alpha_1 t_{21} & \alpha_2 t_{22} \end{bmatrix}$$

From this we have $\alpha_2 t_{21} = \alpha_1 t_{21}$ and $t_{21} = 0$. Moreover, $t_{12} = at_{22}/(\alpha_2 - \alpha_1)$.

If

$$T^{-1}BT = \begin{bmatrix} \beta_2 & 0 \\ 0 & \beta_1 \end{bmatrix}$$

we obtain that $\beta_2 t_{22} = t_{22}\beta_1$. Since $\beta_1 \neq \beta_2$, we have $t_{22} = 0$, which contradicts the non-singularity of T. Hence $T^{-1}BT$ must be

$$\begin{bmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{bmatrix}$$

In a similar fashion, we can prove that if

$$T^{-1}AT = \begin{bmatrix} \alpha_2 & 0 \\ 0 & \alpha_1 \end{bmatrix}$$

then

$$T^{-1}BT = \begin{bmatrix} \beta_2 & 0 \\ 0 & \beta_1 \end{bmatrix}$$

This completes the proof. □

Proof of Theorem 4.4: It is enough to prove the sufficiency. As the matrices $H(1)$, $H(2)$, ..., $H(N)$ pairwise commute, from Theorem 4.3, there exists a unitary matrix U, such that

$$U^*H(j)U = \begin{bmatrix} \alpha_j & b_j \\ 0 & \beta_j \end{bmatrix} \quad (1 \leqslant j \leqslant N) \tag{4.1}$$

Without loss of generality, we assume that for each $j \in \underline{N}$, $H(j)$ is an upper triangular matrix.

Group (4.1) into the following three sets

$$\text{Set 1: } \begin{bmatrix} \alpha_1 & b_1 \\ 0 & \alpha_1 \end{bmatrix}, \ldots, \begin{bmatrix} \alpha_i & b_i \\ 0 & \alpha_i \end{bmatrix}, \quad b_r \neq 0, \quad 1 \leqslant r \leqslant i$$

$$\text{Set 2: } \begin{bmatrix} \alpha_{i+1} & b_{i+1} \\ 0 & \beta_{i+1} \end{bmatrix}, \ldots, \begin{bmatrix} \alpha_{i+j} & b_{i+j} \\ 0 & \beta_{i+j} \end{bmatrix}, \quad \alpha_{i+r} \neq \beta_{i+r}, b_{i+r} \neq 0, 1 \leqslant r \leqslant j$$

$$\text{Set 3: } \begin{bmatrix} \alpha_{i+j+1} & 0 \\ 0 & \beta_{i+j+1} \end{bmatrix}, \ldots, \begin{bmatrix} \alpha_N & 0 \\ 0 & \beta_N \end{bmatrix}$$

Let us examine each of the following three possibilities.

Case 1: if Set 1 is not empty, then from (ii) of Lemma 4.3, we know that Set 2 is empty, and the matrices in Set 3 are just scalar multiples of an identity matrix, i.e. $\alpha_{i+1} = \beta_{i+1}, \ldots, \alpha_N = \beta_N$. Thus, we have ($q_j$ is as defined before)

$$\begin{aligned} x_k &= H(1)^{q_1} H(2)^{q_2} \ldots H(N)^{q_2} x_0 \\ &= \begin{bmatrix} \alpha_1 & b_1 \\ 0 & \alpha_1 \end{bmatrix}^{q_1} \cdots \begin{bmatrix} \alpha_i & b_i \\ 0 & \alpha_i \end{bmatrix}^{q_i} \alpha_{i+1}^{q_{i+1}} \ldots \alpha_N^{q_N} x_0 \\ &= \begin{bmatrix} \alpha_1^{q_1} \ldots \alpha_i^{q_i} \alpha_{i+1}^{q_{i1}} \ldots \alpha_N^{q_N} & f(q_1, q_2, \ldots, q_N) \alpha_1^{q_1} \ldots \alpha_N^{q_N} \\ 0 & \alpha_1^{q_1} \alpha_2^{q_2} \ldots \alpha_N^{q_N} \end{bmatrix} x_0 \end{aligned}$$

where $f(q_1, q_2, \ldots, q_N)$ is a finite degree polynomial in $q_1, q_2, \ldots, q_N$. Thus, from the condition of the proposition and the Law of Large Numbers, we obtain that $x_k \overset{k \to \infty}{\to} 0$ almost surely, i.e. the system (2.1) is almost surely stable.

Case 2: If Set 1 is empty, and Set 2 is not empty, then the matrices in Set 2 can be simultaneously diagonalized. Furthermore, from (iii) of Lemma 4.3, there is a T so that

$$T^{-1} \begin{bmatrix} \alpha_i & b_i \\ 0 & \beta_i \end{bmatrix} T = \begin{bmatrix} \alpha_i & 0 \\ 0 & \beta_i \end{bmatrix} \quad (1 \leqslant i \leqslant j)$$

Also, note that if Set 2 is not empty, then the matrices in Set 3 are again scalar multiples of the identity matrix. Therefore, we have

$$\begin{aligned} x_k &= \begin{bmatrix} \alpha_1 & b_1 \\ 0 & \beta_1 \end{bmatrix}^{q_1} \cdots \begin{bmatrix} \alpha_j & b_j \\ 0 & \beta_j \end{bmatrix}^{q_j} \alpha_{j+1}^{q_{j+1}} \ldots \alpha_N^{q_N} x_0 \\ &= T \begin{bmatrix} \alpha_1^{q_1} \alpha_2^{q-2} \ldots \alpha_N^{q_N} & 0 \\ 0 & \beta_1^{q_1} \beta_2^{q-2} \ldots \beta_N^{q_N} \end{bmatrix} T^{-1} x_0 \end{aligned} \tag{4.2}$$

In deriving (4.2), we have used the fact that $\alpha_{j+1} = \beta_{j+1}, \ldots, \alpha_N = \beta_N$. From (4.2), it follows that under the condition of the proposition, the Law of Large Numbers guarantees that $x_k \overset{k \to \infty}{\to} 0$. This implies that system (2.1) is almost surely stable.

Case 3: if Set 1 and Set 2 are empty, then $H(1)$, $H(2)$, ..., $H(N)$ are diagonal. It is easy to see that in this case, system (2.1) is almost surely stable. □

5. Concluding remarks

In this paper, we have studied stability problems of discrete-time jump linear systems and we have obtained a set of stability criteria both for almost-sure stability and δ-moment stability, which are easy to check in applications. Some of the criteria developed turn out to be necessary and sufficient conditions for almost-sure stability and δ-moment stability for one dimensional systems. We also studied the relationship between δ-moment stability and almost-sure stability. It is shown that for scalar systems, δ-moment stability implies almost-sure stability. Also, for $\delta > 0$ sufficiently small, the region for δ-moment stability is a good estimate of the region for almost-sure stability. This suggests that the almost sure stability information is somehow contained in the δ-moment stability information. For systems with a special commuting structure, some necessary and sufficient conditions for almost sure stability have been obtained.

We mention again that although we deal with systems with an i.i.d. form process exclusively in this paper, all the results have their analogues in systems with a Markov form process. Much future research work can be identified. The most interesting is probably the further investigation of the relationship between almost-sure and δ-moment stability. We expect that results hold in the higher dimensional cases that are similar to those obtained for one-dimensional systems in this paper. Moreover, a weaker testable (necessary and) sufficient condition for δ-moment stability for general jump linear systems is also needed to achieve a significant stability criteria for engineering applications. Of course, jump linear systems are only a small family of stochastic systems. One may ask for similar results for a more general family of systems. In a subsequent paper, we will report on further research results in these directions.

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Appendix

Proof of Lemma 4.1: We only need to prove the 2-norm case because any norms on $\mathcal{R}^{n\times n}$ are equivalent.

Using Jordan's theorem, there exists a non-singular T such that

$$TAT^{-1} = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_r \end{bmatrix} \stackrel{\text{def}}{=} J, \quad \text{where } J_i = \begin{bmatrix} \alpha_i & 1 & & \\ & \alpha_i & 1 & \\ & & \ddots & 1 \\ & & & \alpha_i \end{bmatrix}.$$

Thus, we have $A^k = T^{-1}J^kT$. Since

$$\begin{aligned} \|A\| &= \|T^{-1}J^kT\| \leqslant (\|T\|\cdot\|T^{-1}\|)\|J^k\| \\ &\leqslant (\|T\|\cdot\|T^{-1}\|)\max_{1\leqslant l\leqslant r}\{\|J_l^k\|\} \end{aligned}$$

Therefore, it is enough to show that the result holds for any Jordan block J in the form

$$J = \begin{bmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}_{m\times m}$$

Since

$$J^k \begin{bmatrix} \lambda^k & k\lambda^{k-1} & \binom{k}{2}\lambda^{k-2} & \cdots & \binom{k}{m-1}\lambda^{k-m+1} \\ & \lambda^k & k\lambda^{k-1} & \cdots & \binom{k}{m-2}\lambda^{k-m+2} \\ & & \ddots & & \vdots \\ & & & & \lambda^k \end{bmatrix}$$

it follows that

$$\begin{aligned} \|J^k\| &= \sup_{\|x\|=1} (x'(J')^k J^k x)^{1/2} \\ &= \sup_{\|x\|=1} \left[\sum_{i,j} x_i x_j (P_m^{ij}(k))(\lambda^{k-m})^2\right]^{1/2} \\ &\leqslant \sup_{\|x\|=1} \left[\sum_{i,j} |x_i x_j|\, P_m^{i,j}(k)(|\lambda|/\alpha)^{2(k-m)}\right]^{1/2} \alpha^{k-m} \end{aligned} \qquad \text{(A 1)}$$

Where, in (A 1), $P_m^{i,j}(k)$ is a polynomial of k with degree less than $2m$. Since $|\lambda|/\alpha < 1$ for any eigenvalue λ of A, we have for any $1 \leqslant i,\ j \leqslant m$

$$P_m^{i,j}(k)(|\lambda|/\alpha)^{2(k-m)} \to 0,\ k \to \infty$$

Thus, we conclude that $\exists\, M > 0$, such that

$$\|A^k\| \leqslant M\alpha^k, \quad \forall\, k > 0 \qquad \square$$

Remark: In this lemma, we cannot improve the value of α. In fact, we cannot choose α as the spectral radius of A. For example

$$A = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}, \quad A^k = \begin{bmatrix} a^k & ka^{k-1} \\ 0 & a^k \end{bmatrix}$$

Thus, $\lambda_{\max}(A^k(A^k)') = \frac{1}{2}(2 + k^2a^{-2} + ka^{-2}\sqrt{4a^2 + k^2})a^{2k} = f(k)a^{2k}$, i.e. $\|A^k\| = \sqrt{f(k)}a^k$. As we can see, $f(k)$ is unbounded. $\square$

Proof of Lemma 4.2: (i) follows from a direct computation and (iii) follows from (ii). We only show (ii) here. First, assume that A and B are in the Jordan canonical forms as in (ii). We want to show that they can be rewritten in the required forms. We start with J_1 and $\bar{J}_1$.

Case 1: $\dim J_1 > \dim \bar{J}_1$. For $\dim \bar{J}_1$, then J_1 and $\bar{J}_1$ are real Jordan blocks. Let

$$J_1 = \begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}_{p\times p}, \quad \bar{J}_1 = \begin{bmatrix} \mu & 1 & & \\ & \mu & \ddots & \\ & & \ddots & 1 \\ & & & \mu \end{bmatrix}_{q\times q}$$

and let J be the $p \times p$ submatrix of B on the upper left, i.e.

$$J = \begin{bmatrix} \bar{J}_1 & 0 \\ 0 & \Delta_1 \end{bmatrix}$$

also let

$$J_1 = \begin{bmatrix} \delta_1 & \Delta_2 \\ 0 & \delta_2 \end{bmatrix}, \quad \delta_1 = \begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}_{q\times q}$$

$$\delta_2 = \begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}_{(p-q)\times(p-q)}, \quad \Delta_2 = \begin{bmatrix} 0 & & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{bmatrix}_{q\times(p-q)}$$

Thus, from the $p \times p$ upper left submatrices of AB and BA and the commutativity of A and B, we have $\Delta_2\Delta_1 = \bar{J}_1\Delta_2$, and if $p - q \geqslant 1$, this gives

$$\begin{bmatrix} 0 & & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} \sigma & * & \cdots & * \\ & \sigma & \ddots & \vdots \\ & & \ddots & * \\ & & & \sigma \end{bmatrix} = \begin{bmatrix} \mu & 1 & \cdots & 0 \\ & \mu & \ddots & \vdots \\ & & \ddots & 1 \\ & & & \mu \end{bmatrix} \begin{bmatrix} 0 & & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{bmatrix}$$

i.e.

$$\begin{bmatrix} 0 & & \cdots & * \\ \vdots & & & \vdots \\ 0 & & \cdots & 0 \\ \sigma & * & \cdots & * \end{bmatrix} = \begin{bmatrix} 0 & & \cdots & * \\ \vdots & & & \vdots \\ 1 & & \cdots & * \\ \mu & * & \cdots & * \end{bmatrix}$$

Comparing the first column on both sides, we obtain that this is not true. Hence $\dim \bar{J}_1 > 1$ is not true.

For $\dim \bar{J}_1 = 1$, then from $AB = BA$, one can observe the upper left $p \times p$ submatrix. It is easy to check that the next $p - 1$ dimensional matrix (i.e. the $J(2, 3, \ldots, p)$ which is the minor matrix including the elements in $2, 3, \ldots, p$ rows and $2, 3, \ldots, p$ columns) is diagonal, this implies that $\bar{J}_2, \bar{J}_3, \ldots, \bar{J}_p$ are of dimension one. This can be seen by observing the $(1, 3)$ elements both in AB and BA together with the commutativity. Furthermore, from (i), $\bar{J}_1 = \bar{J}_2 = \cdots = \bar{J}_p$. In this way, we can rewrite $\bar{J}_1$ as the $p \times p$ submatrix at the upper left position in B. Thus, $\dim J_1 = \dim \bar{J}_1$, and they are at the same position in A and B, respectively.

If $\dim J_1 < \dim \bar{J}_1$, we can perform the same procedure. Continuing this reconstruction, we can obtain our D–J canonical form. This completes our proof.

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