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Sufficient conditions for the stability of interval matrices

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The stability of interval dynamical systems is studied. Sufficient conditions for the polytope of interval matrices are examined and some of the proofs are greatly simplified. More generally sufficient conditions are obtained and the approach taken in the new proofs has the potential for further generalizations of the result obtained in this paper.

1. Introduction

After the publication of Kharitonov's (1978) paper, many researchers have studied the stability of interval dynamical systems. Motivated by this, the stability of interval matrices has been recently investigated. Bialas (1983) generalized Kharitonov's result to this case and claimed that a necessary and sufficient condition for the stability of interval matrices is the stability of the vertex matrices. This was shown to be incorrect by Barmish and Hollot (1984), and Barmish *et al*. (1988) began to consider the stability of the convex hull (polytope) of matrices and provided counterexamples to some 'tempting' conjectures for sufficient conditions for the stability of the polytope of matrices. By specifying the vertices, many researchers have obtained necessary and sufficient conditions for the stability of a polytope of matrices. Shi and Gao (1986) and Soh (1990 a) found that when the vertex matrices are symmetric, the polytope of matrices is stable if and only if the vertex matrices are stable. Wang (1991) generalized this to the case when the vertex matrices are normal. Liao (1987 a) proved that this is also true for Metzler vertex matrices. But all these criteria are too conservative to be useful in many practical applications. Many researchers have attempted to seek sufficient conditions which are less conservative. Jiang (1987) obtained a sufficient condition which stated that if the symmetric components of the vertex matrices are stable, then the polytope is stable. Argoun (1986) used the Gershgorin's circle theorem to obtain a sufficient condition, which was shown to be false by Xu and Shao (1989), and Soh (1990 b) corrected this result by using the matrix measure. Also using the matrix measure Fang, *et al*. (1992) obtained a very general sufficient condition for the stability of the polytope, which generalized all the above criteria. There is, however, another approach to this problem. Similar to the study of the stability of ordinary differential equations, the comparison principle of Lakshmikantham and Leela (1969) can be used. Xu (1985) obtained a sufficient condition by transforming the vertex matrices into M-matrices. Liao (1987 b) generalized the result for application to large-scale systems. This result is important because when the dimension of the dynamical systems is very large, the result can be

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used to reduce the problem to the study of lower dimensional systems, which makes the approach computationally attractive.

In this paper, we comment on some of the previous results, generalize some of these results and at the same time simplify their proofs. Some additional sufficient conditions are obtained and some necessary conditions are provided.

2. Main contributions

Let the non-negative cone of real matrices $A_1, A_2, \ldots, A_p$ be defined as

$$W = \left\{A \mid A = \sum_{i=1}^{p} \beta_i A_i,\ \beta_i \geq 0,\ i = 1, 2, \ldots, p\right\}$$

and the convex hull (polytope) as

$$S = \left\{A \mid A = \sum_{i=1}^{p} \alpha_i A_i,\ \alpha_i \geq 0\ (1 \leq i \leq p),\ \sum_{i=1}^{p} \alpha_i = 1\right\}$$

Jiang (1987) proved that if the symmetric components $(A_i + A_i^{\mathrm{T}})/2$ of vertex matrices are stable, then the cone W is stable. Although some of details in the proof are not necessary, which has already been pointed out, a simplification of the proof can be obtained. Mansour (1988) proposed a short proof by using Liapunov's second method and generalized the above result, i.e., if P is positive definite, and $PA_i + A_i^{\mathrm{T}}P$ $(1 \leq i \leq p)$ are stable, then the cone W is stable. Let $|x|$ denote a vector norm of x on $\mathbb{C}^n$, and $\|A\|$ the induced matrix norm of the A by the vector norm $|\cdot|$. The matrix measure $\mu(A)$ of the matrix A induced by the above matrix norm (or say, the vector norm) is defined by

$$\mu(A) \triangleq \lim_{\theta \downarrow 0+} \frac{\|I + \theta A\| - 1}{\theta}$$

where I is the identity matrix. The properties of matrix measure can be found in Desoer and Vidyasager (1975). Let $\mu_2(A)$ denote the matrix measure induced by the vector 2-norm or the Euclidean norm: we propose the following simple proof just based on Jiang's approach (in what follows, we use $\mathbb{R}$ to denote the real part of a complex number, and $\lambda(A)$ or $\lambda_i(A)$ to denote the eigenvalues of the matrix A).

Proof: It is easy to see that for any $c \geq 0$ and matrices A and B, we have $\mu_2(A + B) \leq \mu_2(A) + \mu_2(B)$ and $\mu_2(cA) = c\mu_2(A)$. From this we obtain

$$\mu_2\left(\sum_{i=1}^{p} \beta_i A_i\right) \leq \sum_{i=1} \beta_i \mu_2(A_i)$$

Thus if the matrices $(A_i + A_i^{\mathrm{T}})/2$ $(1 \leq i \leq \mathrm{p})$ are stable, then for any $A \in W$, $A \neq 0$ matrix,

$$\mu_2(A) = \mu_2\left(\sum_{i=1}^{p} \beta_i A_i\right) \leq \sum_{i=1}^{p} \beta_i \mu_2(A_i) = \sum_{i=1}^{p} \beta_i \max_i \lambda_i\left(\frac{A_i + A_i^{\mathrm{T}}}{2}\right) < 0$$

Hence $\max_i \mathbb{R}\lambda_i(A) < 0$, which means that A is stable and that the cone W is stable. □

This procedure can be generalized to a more general case, we have obtained the following result.

Theorem 1 (Fang *et al.* 1992): *If there exists a matrix measure μ such that $\mu(A_i) < 0$ $(1 \leq i \leq p)$, then the cone W is stable. (We just regard the zero matrix in the cone as an exceptional element).*

This result is more general than Jiang's and Mansour's because the matrix measure depends upon the choice of the vector norm on R^n. If we choose the 2-norm, we obtain Jiang's result as shown above. If P is positive definite, let $P = Q^2$, then when we use the norm $\|x\| = \|Qx\|_2$, we obtain Mansour's generalization of Jiang's result. In fact, if we use μ_Q to be the induced matrix measure by the above norm, we have

$$\mu_Q(A) = \mu_2(QAQ^{-1}) = \max_i \mathbb{R}\lambda_i \left(\frac{QAQ^{-1} + Q^{-\mathrm{T}}A^{\mathrm{T}}Q^{\mathrm{T}}}{2} \right)$$
$$= \max_i \mathbb{R}\lambda_i \left(Q^{-1} \frac{PA + A^{\mathrm{T}}P}{2} Q^{-1} \right) \leq \frac{\max_i \lambda_i(PA + A^{\mathrm{T}}P)}{2\alpha(P, A)}$$

where $\alpha(P, A) = \max_i \lambda_i(P)$ if $PA + A^{\mathrm{T}}P$ is negative semidefinite and $\alpha(P, A) = \min_i \lambda_i(P)$ if $PA + A^{\mathrm{T}}P$ is positive semidefinite. From this we can prove our claim. In fact, both Jiang's and Mansour's results essentially require that a positive definite solution exists for a simultaneous set of Lyapunov equations formed by the vertex matricies, this is apparently too conservative. Our criteria may provide a relaxed sufficient condition. As we pointed out in Fang *et al.* (1992), when the 1-norm and the ∞-norm are used, we can obtain Argoun's result (Argoun 1986), and Soh's result (1990) as special cases. Wang's result (Wang 1991) can be deduced from Theorem 1 in the following way: If S is stable, then the vertex matrices of S are obviously stable. Suppose that the vertex matrices of S, i.e., $A_1, A_2, \ldots, A_p$, are stable. Note, for any normal matrix A, from Theorem 2.5.4 on p. 101 of Horn and Johnson (1985), there exists a unitary matrix U such that $A = UDU^{-1}$ where D is a diagonal matrix. Using the vector 2-norm, which is invariant under a unitary transformation, the induced matrix measure is also invariant, and we have

$$\mu_2(A) = \mu_2(UDU^{-1}) = \mu_2(D) = \max_i \mathbb{R}\lambda_i(D) = \max_i \mathbb{R}\lambda_i(A)$$

Hence, if A_i is normal and stable, then $\mu_2(A_i) < 0$, and from Theorem 1, we conclude that S is stable, which completes the proof. The above discussion illustrates that Theorem 1 gives a very general sufficient condition for interval matrices.

The matrix measure can also be used to study the robustness of interval dynamical systems. The following theorem is obvious and useful.

Theorem 2: *For any matrix measure μ, and any constant c, $\mu(A) \leq c$ for any $A \in S$ if and only if $\mu(A_j) \leq c$ $(1 \leq j \leq \mathrm{p})$. Moreover, the set*

$$S_c = \{A | \mu(A) \leq c\}$$

is convex.

From this theorem, we can obtain some results on the robustness of dynamical systems with parameter perturbations.

Corollary 1 (Fang *et al.* 1992): *For the system described by*

$$\dot{x}(t) = (A + \Delta A)x(t)$$

where ΔA represents the parameter uncertainty, if there exists a matrix measure μ such that $\mu(\Delta A) < -\mu(A)$, then the system is asymptotically stable, i.e. the system is always asymptotically stable for all parameter uncertainty ΔA satisfying $\mu(\Delta A) < -\mu(A)$.

From Lyapunov's theorem, A is stable if and only if there exists a positive definite matrix P, and a matrix measure μ such that $\mu(A^TP + PA) < 0$. In the above corollary, if $\Delta A \in \text{conv}\{E_i, i = 1, 2, \ldots, p\}$, the convex hull (polytope) of matrices $E_1, E_2, \ldots, E_p$, then we obtain

Corollary 2: *Suppose for any positive matrix Q, P is the positive definite solution to the Lyapunov equation $PA + A^Tp = -Q$, then the system is asymptotically stable if there exists a matrix measure μ, so that $\mu(E_i^TP + PE_i) < -\mu(-Q)$ $(1 \leq i \leq p)$.*

Proof: $A + \Delta A = A + \Sigma_{i=1}^{p} \alpha_i E_i$, so

$$\begin{aligned}\mu((A + \Delta A)^TP + P(A + \Delta A)) &= \mu(A^TP + PA + \Delta A^TP + P\Delta A) \\ &= \mu\left(-Q + \sum_{i=1}^{p} \alpha_i(E_i^TP + PE_i)\right)\end{aligned}$$

Thus from Theorem 2 and the above corollary, if the condition in the corollary holds, then $\mu(A + \Delta A) < 0$, which implies that the system is asymptotically stable. □

If we choose $Q = 2I$, we obtain Foo and Soh's result (1990).

Corollary 3: *If P is the solution of the Lyapunov equation $A^TP + PA = -2I$, then the system is asymptotically stable if $\mu(E_i^TP + PE_i) < 2$ $(1 \leq i \leq p)$.*

A necessary condition for the non-negative cone W to be unstable is presented in Jiang (1987) and Mansour (1988); we point out that they are too conservative. Jiang's result (Theorem 2 in Jiang (1987)) states that W is not asymptotically stable if $\lambda_{\min}(B_i) \geq 0$ for some i, where $B_i = (A_j + A_i^T)/2$, the symmetric part of A_i. This in fact requires that B_i is positive semidefinite, which implies that A_i is not stable. Mansour's necessary condition suffers from the same disadvantage. The following result provides a necessary condition which is easier to test.

Theorem 3: *W is not stable if some A_i is not stable.*

Proof: This is obvious because $A_i \in W$. □

Corollary 4: *W is not stable if for some A_i, $\text{tr}(A_i) \geq 0$, where $\text{tr}(A)$ is the trace of A.*

Remark: If $\lambda_{\min}(B_i) \geq 0$ for some i, then $\text{tr}(B_i) = \sum \lambda(B_i) \geq 0$, and $\text{tr}(A_i) = \text{tr}(B_i)$, hence $\text{tr}(A_i) \geq 0$, which also implies the instability of A_i. For Mansour's result, we can show that his condition implies the instability of the vertex matrices. His necessary condition states that if for a positive definite matrix P, $B_i = A_i^TP + PA_i$ is positive semidefinite, then W is not stable. We notice that

$$2\,\mathrm{tr}(A_i) = \mathrm{tr}(A_i + P^{-1}A_i^{\mathrm{T}}P) = \mathrm{tr}(P^{-1}B_i) \geqslant \min\lambda(B_i)\,\mathrm{tr}(P^{-1}) \geqslant 0$$

This implies that A_i is not stable. We may conjecture that if $\mathrm{tr}(PA_i + A_i^{\mathrm{T}}P) \geqslant 0$, then W is not stable. This is not true, for example, let

$$A = \begin{bmatrix} -1 & 8 \\ 0 & -1 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 0{\cdot}5 \\ 0{\cdot}5 & 1 \end{bmatrix}$$

then $\mathrm{tr}(A^{\mathrm{T}}P + PA) = 4 > 0$, but A is stable. □

Next, we pursue the comparison approach. Xu's result (1985) depends on Lemma 2 of Xu (1985), which states the following.

Lemma 1 (Lemma 2 from Xu 1985): *If $A = (a_{ij})$ and $B = (b_{ij})$, and if*

$$a_{ii} \leqslant b_{ii} < 0,\ |a_{ij}| \leqslant b_{ij}\ (i \neq j)\ (i, j = 1, 2, \ldots, n)$$

then stability of B implies the stability of A.

We give a simpler proof here.

Proof: Let $C = (c_{ij})$ be defined as the following:

$$c_{ii} = a_{ii}, \quad c_{ij} = |a_{ij}|\ (i \neq j)$$

Then it is well-known (Šiljak 1978) that A is stable if C is stable. Let α be chosen so that $B = \alpha I + P_1$ and $C = \alpha I + P_2$, where P_1 and P_2 are non-negative matrices. Because $C \leqslant B$ elementwise, we have $P_2 \leqslant P_1$, and we have $\rho(P_2) \leqslant \rho(\mathrm{P}_1)$, where ρ is the spectral radius (Šiljak 1978). Because P_1 and P_2 are non-negative matrices, $\rho(P_1)$ and $\rho(P_2)$ are also eigenvalues of P_1 and P_2, respectively. Suppose that B is stable, then $\alpha + \rho(P_1) < 0$, hence $\alpha + \rho(P_2) < 0$, thus

$$\max_{1\leqslant i\leqslant n} \mathbb{R}\,\lambda_i(C) = \alpha + \max_{1\leqslant i\leqslant n} \mathbb{R}\,\lambda_i(P_2) \leqslant \alpha + \rho(P_2) < 0$$

which implies that C is stable. Therefore, A is also stable. □

Liao (1987 a) also studied a similar problem and obtained a necessary and sufficient condition. In fact, his results can be directly obtained from Lemma 1. Let $N^*(P, Q)$ be the interval matrices, where $P = (p_{ij})$ and $Q = (q_{ii})$ satisfy $q_{ii} < 0$ $(i = 1, 2, \ldots, n)$ and $p_{ij} \geqslant 0$ $(i \neq j,\ i, j = 1, 2, \ldots)$. Liao obtained the result that $N^*(P, Q)$ is stable if and only if Q is stable. We can give the following simple proof by using Lemma 1. The proof of necessity is trivial. For sufficiency, suppose that Q is stable, for any $A \in N^*(F, Q)$, from $p_{ij} \leqslant a_{ij} \leqslant q_{ij}$ and the constraints on P and Q, we have

$$a_{ii} \leqslant q_{ii} < 0, \quad |a_{ij}| = a_{ij} \leqslant q_{ij}$$

which satisfies the condition of Lemma 1, hence A is stable, and so is $N^*(P, Q)$. □

Liao (1987 b) generalized this result to large-scale systems. We first give a very short proof and then present another sufficient condition for the stability of interval dynamical systems.

Let

$$N(P, Q) = \{A = A(a_{ij})_{n\times n} | P(p_{ij}) \leq A(a_{ij}) \leq Q(q_{ij})\}$$
$$A = \operatorname{diag}(A_{11}, A_{22}, \ldots, A_{rr}) + ((1 - \delta_{ij})A_{ij})$$
$$P = \operatorname{diag}(P_{11}, P_{22}, \ldots, P_{rr}) + ((1 - \delta_{ij})P_{ij})$$
$$Q = \operatorname{diag}(Q_{11}, Q_{22}, \ldots, Q_{rr}) + ((1 - \delta_{ij})Q_{ij})$$
$$A_{ij} \in N(P_{ij}, Q_{ij}) \ (i, j = 1, 2, \ldots, r)$$

where A_{ij}, P_{ij}, Q_{ij} are $n_i \times n_i$ matrices, $\sum_{i=1}^{r} n_i = n$, δ_{ij} is the Kronnecker delta function . Define $m_{ij} = \max\{\|A_{ij}\| | A_{ij} \in N(P_{ij}, Q_{ij})\}$.

Theorem 4 (Liao 1987 b): *If for any* $A_{ii} \in N(P_{ii}, Q_{ii})$, *there exist* $M_i > 0$, $\alpha_i > 0$ *such that*

$$\|\exp(A_{ii}(t - t_0))\| \leq M_i \exp(-\alpha_i(t - t_0)), \quad (i = 1, 2, \ldots, r)$$

with

$$b_{ij} = \frac{M_i}{\alpha_i}(1 - \delta_{ij})m_{ij}$$

and if $\rho(B) < 1$, where $\rho(B)$ *is the spectral radius of* $B(b_{ij})$, *then* $N(P, Q)$ *is stable*.

Although it has been pointed out that this result could be proved by M-matrix theory, it is only true for a special case, i.e. for the results in Liao (1987 a). The original proof of Theorem 4 is too complicated, we give the following simple proof.

Proof: Consider the following dynamical system

$$\dot{X}(t) = AX(t) \tag{1}$$

where $A \in N(P, Q)$. This is equivalent to the following set of block equations

$$\dot{X}_i(t) = A_{ii}X_i(t) + \sum_{j=1}^{r}(1 - \delta_{ij})A_{ij}X_j(t) \quad i = 1, 2, \ldots, r \tag{2}$$

For any $t_0 \leq t_1 \leq t_2$, we have

$$X_i(t_2) = \exp(A_{ii}(t_2 - t_1))X_i(t_1) + \int_{t_1}^{t_2} \exp(A_{ii}(t_2 - s)) \sum_{j=1}^{r}(1 - \delta_{ij})A_{ij}X_j(s)\,\mathrm{d}s \tag{3}$$

From this and the condition (i), we can obtain

$$\|X_i(t_2)\| \leq M_i \exp(-\alpha_i(t_2 - t_1))\|X(t_1)\| + \sum_{j=1}^{r} M_i(1 - \delta_{ij})m_{ij}\int_{t_1}^{t_2}\exp(-\alpha_i(t_2 - s))\|X_j(s)\|\,\mathrm{d}s \tag{4}$$

Let $X_i = \overline{\lim}_{t\to\infty}\|X_i(t)\|$, then from (4), we obtain

$$X_i \leq \overline{\lim_{t_1\to\infty}}\ \overline{\lim_{t_2\to\infty}} \sum_{j=1}^{r} b_{ij} \int_{t_1}^{t_2} \alpha_i \exp(-\alpha_i(t_2 - s))\, \|X_j(s)\|\, ds$$

$$\leq \lim_{t_1\to\infty}\left[\overline{\lim_{t_2\to\infty}} \sum_{j=1}^{r} b_{ij} \int_{t_1}^{t_2} \alpha_i \exp(-\alpha_i(t_2 - s))\, ds\right] \lim_{t_2\to\infty} \max_{t_1\leq t\leq t_2} \|X_j(t)\| \leq \sum_{j=1}^{r} b_{ij}X_j$$

Thus, if we let $Y = (X_1, X_2, \ldots, X_r)^{\mathrm{T}}$, then we have

$$0 \leq Y \leq BY$$

From this we obtain $Y \leq B^k Y$, if $\rho(B) < 1$, let k go to ∞, we conclude that $Y = 0$, therefore $X(t) \to 0$ as $t \to \infty$, which means that the dynamical system (1) is stable. This completes the proof. □

From the proof, we can also obtain another sufficient condition as follows.

Theorem 5: *In Theorem* 4, *condition* (ii) *can be replaced by*

(ii)′. *Let* $b_{ij} = M_i(1 - \delta_{ij})m_{ij}$, *and* $\rho(B) < \min\{\alpha_i\}$.

Proof: Similar to the proof of Theorem 4, the following can be obtained

$$\|X_i(t)\| \leq M_i \exp(-\alpha_i(t - t_0))\, \|X_i(t_0)\| + \sum_{j=1}^{r} M_i(1 - \delta_{ij})m_{ij} \int_{t_0}^{t} \exp(-\alpha_i(t - s))\, \|X_j(s)\|\, ds$$

Let $\alpha = \min_{1\leq i\leq r} \alpha_i$, then we have

$$\|X_i(t)\| \leq M_i \exp(-\alpha(t - t_0))\, \|X_i(t_0)\| + \sum_{j=1}^{r} b_{ij} \int_{t_0}^{t} \exp(-\alpha(t - s))\, \|X_j(s)\|\, ds \tag{5}$$

Let $Y_i(t) = \exp(\alpha t)\, \|X_i(t)\|$, and $Y = (Y_1, Y_2, \ldots, Y_r)^{\mathrm{T}}$, then from (5), we obtain

$$Y_i(t) \leq M_i Y_i(t_0) + \sum_{j=1}^{r} b_{ij} \int_{t_0}^{t} Y_j(s)\, ds \tag{6}$$

which is equivalent to

$$Y(t) \leq MY(t_0) + \int_{t_0}^{t} BY(s)\, ds \tag{7}$$

where $M = \mathrm{diag}(M_1, M_2, \ldots, M_r)$. From Theorem 1.9.3 on p. 40 of Lakshmikanthanm and Leela (1969) (a comparison principle from Gronwall–Bellman Lemma), we have

$$Y(t) \leq \exp(Bt)MY(t_0)$$

Let $\bar{X}(t) = (\|X_1(t)\|, \|X_2(t)\|, \ldots, \|X_r(t)\|)^{\mathrm{T}}$, then

$$\bar{X}(t) \leq \exp(-(\alpha I - B)t)\ MY(t_0)$$

Since from (ii)′ $\mathbb{R}\lambda(\alpha I - B) \geq \alpha - \rho(B) > 0$ (because B is a non-negative matrix), this implies that the system (1) is asymptotically stable. So $N(P, Q)$ is stable. □

Remark: The proof of Theorem 5 provides us with another approach to study this problem, and it may be applied to more general case. We will investigate this later. □

3. Conclusions

In this paper, we have discussed some of the sufficient conditions for the stability of interval matrices which have been obtained to date. We have given simplified proofs for some of the results and generalized some of the previously known results. We proposed a new approach to the study of stability of interval matrices implied by our method of proof, this approach may be useful for continued study of more general problem formulations.

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REFERENCES

ARGOUN, M. B., 1986, On sufficient conditions for Stability of interval matrices. *International Journal of Control*, **44**, 1245–1250.

BARMISH, B. R., FU, M. and SALEH, S., 1988, Stability of a polytope of matrices: counterexamples. *IEEE Transactions on Automatic Control*, **33**, 569–572.

BARMISH, B. R., and HOLLOT, C. V., 1984, Counter-example to a recent result on the stability of interval matrices by S. Bialas. *International Journal of Control*, **39**, 1103–1104.

BIALAS, S., 1983, A necessary and sufficient condition for the stability of interval matrices. *International Journal of Control*, **37**, 717–722.

DESOER, C. A., and VIDYASAGAR, M., 1975, *Feedback Systems: Input–Output Properties* (New York: Academic Press).

FANG, Y., LOPARO, K. A., and FENG, X., 1992, A general sufficient condition for the stability of interval matrices. Submitted for publication.

FOO, Y. K. and SOH, Y. C., 1990, Stability analysis of a family of matrices. *IEEE Transactions on Automatic Control*, **35**, 1257–1259.

HORN, R. A., and JOHNSON, C. R., 1985, *Matrix Analysis* (Cambridge, U.K.: Cambridge University Press).

JIANG, C. L., 1987, Sufficient condition for the asymptotic stability of interval matrices. *International Journal of Control*, **46**, 1803–1810.

KHARITONOV, V. L., 1978, Asymptotic stability of an equilibrium position of a family of systems of linear differential equations. *Differential Equations*, **14**(11), 2086–2088.

LAKSHMIKANTHAM, V., and LEELA, S., 1969, *Differential and Integral Inequalities: Theory and Applications*, Vol. I (New York: Academic Press).

LIAO, X. X., 1987 a, Necessary and sufficient conditions for stability of a class of interval matrices. *International Journal of Control*, **45**, 211–214; 1987 b, Stability of interval matrices. *Ibid.*, **45**, 202–210.

MANSOUR, M., 1988, Sufficient conditions for the asymptotic stability of interval matrices. *International Journal of Control*, **47**, 1973–1974.

SHI, Z. C., and GAO, W. B., 1986, A necessary and sufficient condition for the positive-definiteness of interval symmetric matrices. *International Journal of Control*, **43**, 325–328.

ŠILJAK, D. D., 1978, *Large-Scale Dynamic Systems: Stability and Structure* (Amsterdam: North-Holland).

SOH, C. B., 1990 a, Necessary and sufficient conditions for stability of symmetric interval matrices. *International Journal of Control*, **51**, 243–248; 1990 b, Correcting Argoun's approach for the stability of interval matrices. *Ibid.*, **51**, 1151–1154.

WANG, Q., 1991, Necessary and sufficient conditions for stability of a matrix polytope with normal vertex matrices. *Automatica*, **27**, 887–888.

Xu, D. Y., 1985, Simple criteria for stability of interval matrices. *International Journal of Control*, **41**, 289–295.

Xu, S. J., and Shao, C. X., 1989, Comment on 'On sufficient conditions for the stability of interval matrices'. *International Journal of Control*, **49**, 379–381.