Elements of Graph Theory

The aim of this handout is to provide a useful reference for the background of the *graph theory* which will be used as a main mathematical tool in the following topics in our course:

- Network Capacity and Scaling Law
- Network Connectivity (Omni-Directional and Directional)
- Network Formation and Power Control
- Routing Protocols
- Node Deployment and Localization
- Coverage and Boundary Detection

Please make sure you are already familiar with those concepts and theorems before the classes with the topics mentioned above.

The material of this handout comes from the following book:

- P. Santi, *Topology Control in Wireless Ad Hoc and Sensor Networks*, John Wiley and Sons, Chichester, UK, July 2005.

Reference books mentioned in this handout are listed as follows:

- [**Bollobás 1998**] B. Bollobás, *Modern Graph Theory*, Springer-Verlag, New York, 1998.
- [**Goodman and O’Rourke 1997**] J. Goodman and J. O’Rourke, *Handbook of Discrete and Computational Geometry*, CRC Press, New York, 1997.
- [**deBerg et al. 1997**] M. deBerg, M. VanKreveld, M. Overmars and O. Schwarzkopf, *Computational Geometry: Algorithms and Applications*, Springer-Verlag, Berlin Heidelberg, 1997.

A

Elements of Graph Theory

In this Appendix, we report basic definitions and concepts from graph theory that have been used in this book. Most of the material presented in this Appendix is based on (Bollobás 1998) (Section A.1) and on (Goodman and O'Rourke 1997) and (deBerg et al. 1997) (Section A.2).

A.1 Basic Definitions

Definition A.1.1 (Graph) *A graph G is an ordered pair of disjoint sets (N, E), where $E \subseteq N \times N$. Set N is called the vertex, or node, set, while set E is the edge set of graph G. Typically, it is assumed that self-loops (i.e. edges of the form (u, u), for some $u \in N$) are not contained in a graph.*

Definition A.1.2 (Directed and undirected graph) *A graph $G = (N, E)$ is directed if the edge set is composed of ordered node pairs. A graph is undirected if the edge set is composed of unordered node pairs.*

Examples of directed and undirected graphs are reported in Figure A.1. Unless otherwise stated, in the following by *graph* we mean *undirected graph*.

Definition A.1.3 (Neighbor nodes) *Given a graph $G = (N, E)$, two nodes $u, v \in N$ are said to be neighbors, or adjacent nodes, if $(u, v) \in E$. If G is directed, we distinguish between incoming neighbors of u (those nodes $v \in N$ such that $(v, u) \in E$) and outgoing neighbors of u (those nodes $v \in N$ such that $(u, v) \in E$).*

Definition A.1.4 (Node degree) *Given a graph $G = (N, E)$, the degree of a node $u \in N$ is the number of its neighbors in the graph. Formally,*

$$deg(u) = |\{v \in N : (u, v) \in E\}|.$$

If G is directed, we distinguish between in-degree (number of incoming neighbors) and out-degree (number of outgoing neighbors) of a node.

Topology Control in Wireless Ad Hoc and Sensor Networks P. Santi

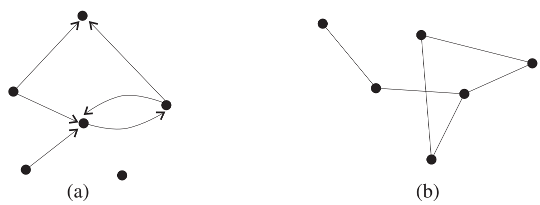


Figure A.1 Examples of directed graph (a) and undirected graph (b).

Definition A.1.5 (Path) *Given a graph $G = (N, E)$, and given any two nodes $u, v \in N$, a path connecting u and v in G is a sequence of nodes $\{u = u_0, u_1, \ldots, u_{k-1}, u_k = v\}$ such that for any $i = 0, \ldots, k-1$, $(u_i, u_{i+1}) \in E$. The length of the path is the number of edges in the path.*

Definition A.1.6 (Cycle) *A cycle is a path $C = \{u_0, \ldots, u_k\}$ such that $k \geq 3$, $u_0 = u_k$, and the other nodes in C are distinct from each other and from u_0.*

Definition A.1.7 (Node distance) *Given a graph $G = (N, E)$ and any two nodes $u, v \in N$, their distance $dist(u, v)$ is the minimal length of a path connecting them. If there is no path connecting u and v in G, then $dist(u, v) = \infty$.*

Definition A.1.8 (Graph diameter) *The diameter of graph $G = (N, E)$ is the maximum possible distance between any two nodes in G. Formally,*

$$diam(G) = \max_{u,v \in N} dist(u, v).$$

Definition A.1.9 (Subgraph) *Given a graph $G = (N, E)$, a subgraph of G is any graph $G' = (N', E')$ such that $N' \subseteq N$ and $E' \subseteq E$. Given any subset N' of the nodes in G, the subgraph of G induced by N' is defined as $G_{N'} = (N', E(N'))$, where $E(N') = \{(u, v) \in E : u, v \in N'\}$, that is, $G_{N'}$ contains all the edges of G such that both endpoints of the edge are in N'.*

Definition A.1.10 (Symmetric sub- and supergraph) *Let $G = (N, E)$ be a directed graph. The symmetric subgraph of G, denoted G^-, is the graph obtained from G by removing all edges such that $(u, v) \in E$ and $(v, u) \notin E$. Formally, $G^- = (N, E^-)$, where $(u, v) \in E^-$ if and only if $(u, v) \in E$ and $(v, u) \in E$. The symmetric supergraph of G, denoted as G^+, is the graph obtained from G by adding the reverse edge to all unidirectional edges in G. Formally, $G^+ = (N, E^+)$, where $(u, v) \in E^+$ if and only if $(u, v) \in E$ or $(v, u) \in E$.*

Definition A.1.11 (Order of a graph) *The order of graph $G = (N, E)$ is the number of nodes in G, that is, the cardinality of set N.*

Definition A.1.12 (Complete graph) *The complete graph $K_n = (N, E)$ of order n is such that $|N| = n$, and $(u, v) \in E$ for any two distinct nodes $u, v \in N$.*

Definition A.1.13 (Sparse graph) *A graph $G = (N, E)$ of order n is sparse if $|E| = O(n)$, that is, if the number of edges in G is linear in n. If a graph is sparse, the average node degree is $O(1)$.*

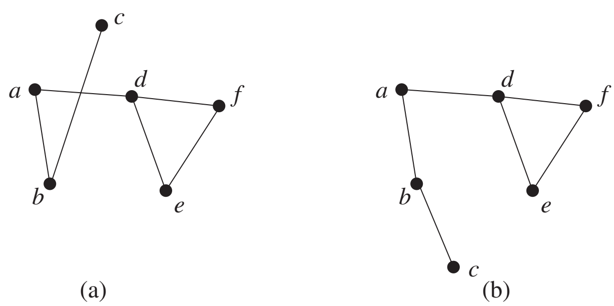


Figure A.2 Notion of graph planarity. The drawing of the graph $G = (\{a, b, c, d, e, f\}, \{(a, b), (b, c), (a, d), (d, e), (d, f), (e, f)\})$ in (a) is not planar; yet, graph G is planar, as shown by the drawing in (b).

Definition A.1.14 (Planar graph) *A graph $G = (N, E)$ is planar if it can be drawn in the plane in such a way that no two edges in E intersect.*

Note that a graph G can be drawn in several different ways; a graph is planar if there exists at least one way of drawing it in the plane in such a way that no two edges cross each other (see Figure A.2).

Definition A.1.15 (Cubic graph) *A graph $G = (N, E)$ is cubic if all its nodes have degree 3.*

Definition A.1.16 (Connected and strongly connected graph) *A graph $G = (N, E)$ is connected if for any two nodes $u, v \in E$ there exists a path from u to v in G. If G is directed, we say that G is strongly connected if for any two nodes $u, v \in E$ there exist a path from u to v, and a path from v to u in G.*

Definition A.1.17 (k-connected and k-edge-connected graph) *A graph $G = (N, E)$ is k-(node-)connected, for some $k \geq 2$, if removing any $k - 1$ nodes from the graph does not disconnect it. Similarly, G is k-edge-connected, for some $k \geq 2$, if removing any $k - 1$ edges from the graph does not disconnect it.*

It can be easily proven that a graph is k-connected if and only if there exist at least k node-disjoint paths between any pair of distinct nodes in G. Similarly, a graph is k-edge-connected if and only if there exist at least k edge-disjoint paths between any pair of distinct nodes in G.

Definition A.1.18 (Graph connectivity and edge connectivity) *The (node) connectivity of a graph $G = (N, E)$, denoted as $\kappa(G)$, is the maximum value of k such that G is k-connected. Similarly, the edge connectivity of G, denoted as $\lambda(G)$, is the maximum value of k such that G is k-edge-connected.*

Theorem A.1.19 *Given a graph $G = (N, E)$, and denoting by $deg_{\min}(G)$ the minimal degree of the nodes in N, we have:*

$$\kappa(G) \leq \lambda(G) \leq deg_{\min}(G).$$

Definition A.1.20 (Weighted graph) *A weighted graph is a graph in which edges, or nodes, or both, are labeled with a weight.*

Definition A.1.21 (Minimum-cost biconnectivity) *A weighted graph $G = (N, E)$ is minimum-cost biconnected if and only if for any node pair $u, v \in N$ there exists a path connecting u and v in the subgraph G' of G obtained by removing all the nodes in $MP - \{u, v\}$, where MP is the path of minimum cost connecting u and v in G.*

Definition A.1.22 (Monotone graph property) *A certain property $\mathcal{P}$ of a graph is said to be monotone if the fact that $\mathcal{P}$ is satisfied in G implies that $\mathcal{P}$ is satisfied in any supergraph G' of G obtained by adding some edges to G.*

An example of monotone graph property is connectivity: if a certain graph G is connected, then any graph G' obtained from G by adding some edges is also connected.

Definition A.1.23 (Dominating set) *Given a graph $G = (N, E)$, a dominating set for G is a set D of nodes such that for any $u \in N - D$ there exists $v \in D$ such that $(u, v) \in E$, that is, any node in the graph is either in D or adjacent to at least one node in D.*

Definition A.1.24 (Connected dominating set) *Given a graph $G = (N, E)$ and a dominating set D for G, D is said to be a connected dominating set if G_D is connected, that is, if the subgraph of G induced by node set D is connected.*

The examples reported in Figure A.3 clarify the notion of dominating set and connected dominating set.

Definition A.1.25 (Tree) *A tree $T = (N, E)$ is a connected graph with n nodes and $n - 1$ edges, that is, a tree is a minimally connected graph.*

Definition A.1.26 (Rooted tree) *A rooted tree $T = (N, E)$ is a tree in which one of the nodes is selected as the tree root. Once the root node r is chosen, the other nodes in the tree can be classified as either internal node or leaf node. An internal node u is such that there exists $v \in N$ such that $(u, v) \in E$ and $dist(u, r) < dist(v, r)$. A leaf node l is such that, for any $v \in N$ such that $(l, v) \in E$, we have $dist(l, r) > dist(v, r)$.*

Definition A.1.27 (Spanning tree) *Given a connected graph $G = (N, E)$, a spanning tree of G is a tree $T = (N, E_T)$ that contains all the nodes in G and is such that $E_T \subseteq E$.*

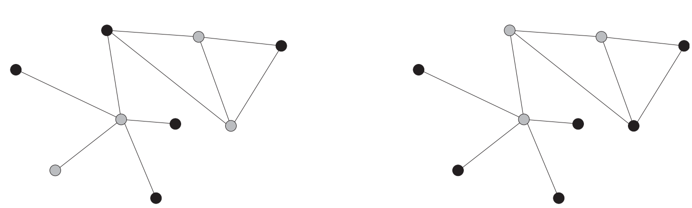

(a) (b)

Figure A.3 Examples of dominating set (a) and connected dominating set (b). The nodes in the dominating set are represented in light gray.

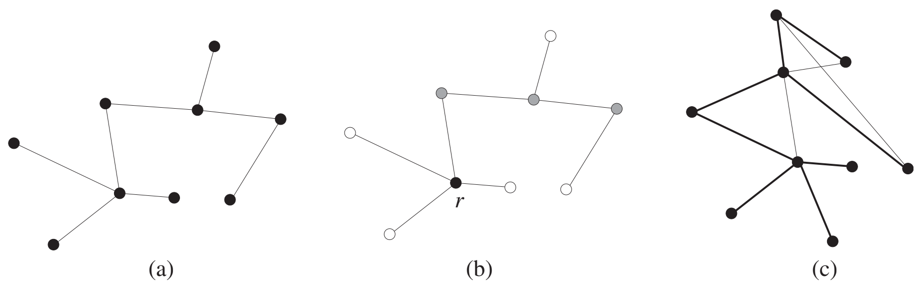


Figure A.4 Examples of tree (a), rooted tree (b), and spanning tree (c). In the rooted tree, internal nodes are gray and leaf nodes are white. The spanning tree on the right is formed by the bold edges.

Figure A.4 reports examples of a tree, a rooted tree, and a spanning tree.

Definition A.1.28 (Cost of a spanning tree) *Given an edge-weighted graph $G = (N, E)$, the cost of a spanning tree T of G is the sum of the weights on its edges.*

Definition A.1.29 (Minimum spanning tree) *Given an edge-weighted graph $G = (N, E)$, a Minimum Spanning Tree (MST) for G is a spanning tree of G of minimum cost.*

Definition A.1.30 (Euclidean MST) *Given a set N of nodes placed in the d-dimensional space (with $d = 1, 2, 3$), and a set of edges E between these nodes, a Euclidean MST (EMST) is a MST of the edge-weighted graph $G = (N, E)$, where each edge has a weight equal to the Euclidean distance between its endpoints.*

Definition A.1.31 (Communication graph) *Given a set N of nodes (representing units of an ad hoc or sensor network), the communication graph is the directed graph $G = (N, E)$ such that edge $(u, v) \in E$ only if v is within u's transmitting range at the current transmit power level.*

Definition A.1.32 (Maxpower graph) *Given a set N of nodes (representing units of an ad hoc or sensor network), the maxpower graph is the communication graph $G = (N, E)$ such that $(u, v) \in E$ if and only if v is within u's transmitting range at maximum power, that is, the maxpower graph contains all possible wireless links between the nodes in the network.*

A.2 Proximity Graphs

Proximity graphs are a class of graphs introduced in the theory of Computational Geometry that are based on proximity relationships between nodes.

Definition A.2.1 (K-neighbors graph) *Given a set N of points in the d-dimensional space, with $d = 1, 2, 3$, and an integer $k \geq 1$, the k-neighbors graph is the directed graph $G_k = (N, E_k)$, where $(u, v) \in E_k$ if and only if v is one of the k closest neighbors of node u.*

Definition A.2.2 (Maximal planar subdivision) *Given a set N of points in the plane, a maximal planar subdivision of N is a planar graph $G = (N, E)$ such that no edge connecting two nodes in N can be added to E without compromising graph planarity.*

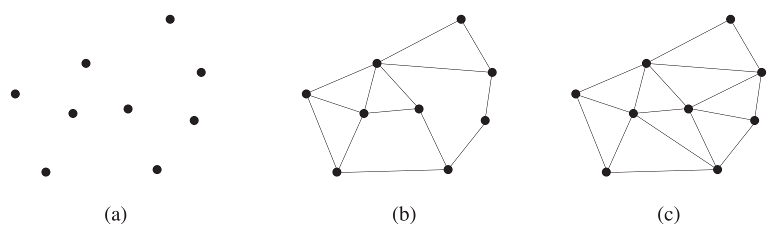


Figure A.5 Examples clarifying the notion of triangulation. In (a) we have a node set N. The graph (b) is a planar subdivision of N, but it is not maximal: in fact, more edges can be added to the graph without compromising planarity. The graph (c) is a triangulation of N.

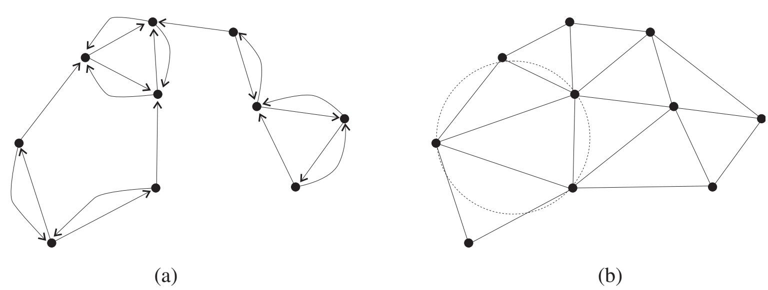


Figure A.6 K-neighbors graph of parameter $k = 2$ (a), and Delaunay triangulation (b). In the Delaunay triangulation, the circumcircle of every triangle (dashed circle) contains no nodes in its interior.

Definition A.2.3 (Triangulation) *Given a set N of points in the plane, a triangulation of N is a maximal planar subdivision whose node set is N.*

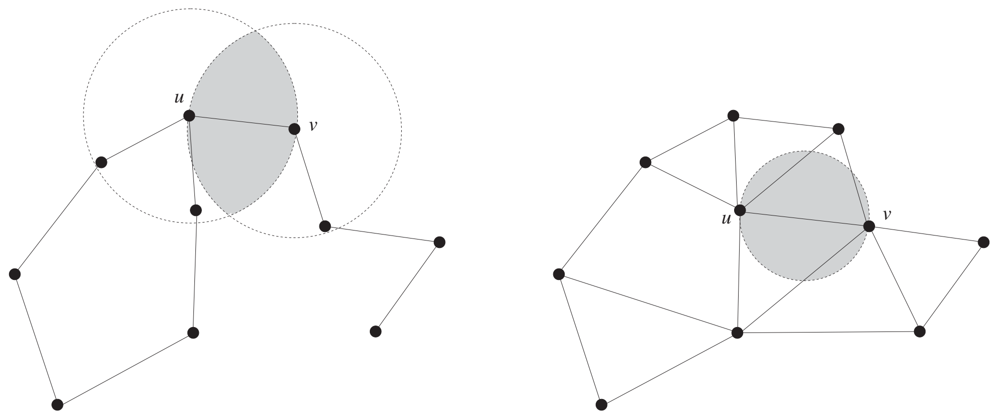
Figure A.5 clarifies the notion of triangulation of a set of points.

Definition A.2.4 (Delaunay triangulation) *Given a set N of points in the plane, the Delaunay triangulation of N is the unique triangulation DT of N such that the circumcircle of every triangle contains no points of N in its interior.*

The k-neighbors graph and Delaunay triangulation of a set of points in the plane are reported in Figure A.6.

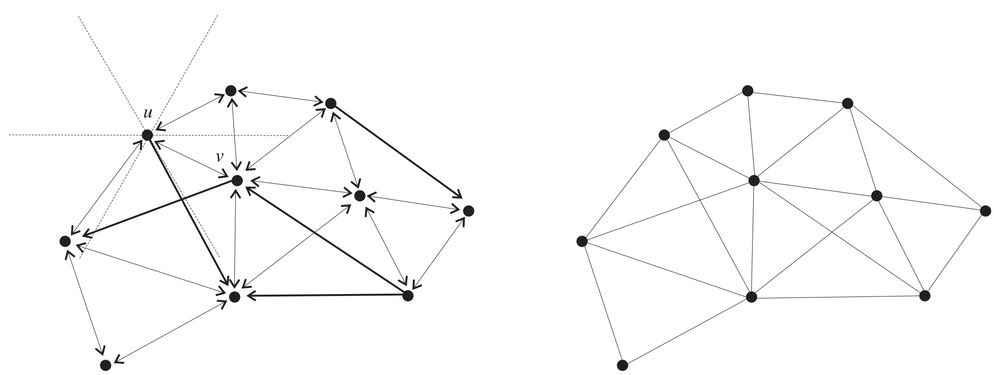
Definition A.2.5 (Relative neighborhood graph) *Given a set N of points in the plane, the Relative Neighborhood Graph (RNG) of N is the graph $RNG = (N, E)$ such that $(u, v) \in E$ if and only if $lune(u, v)$ does not contain any other point of N in its interior, where $lune(u, v)$ denotes the moon-shaped region formed as the intersection of the two circles of radius $\delta(u, v)$ centered at u and at v.*

Definition A.2.6 (Gabriel graph) *Given a set N of points in the plane, the Gabriel Graph (GG) of N is the graph $GG = (N, E)$ such that $(u, v) \in E$ if and only if the circle that has segment $\overline{uv}$ as diameter does not contain any other point of N in its interior.*



(a) (b)

Figure A.7 Relative Neighborhood Graph (a) and Gabriel Graph (b). In the RNG, edge (u, v) exists if and only if $lune(u, v)$ (shaded region) is empty. In the GG, edge (u, v) exists if and only if the circle that has segment $\overline{uv}$ as diameter (shaded region) is empty.



(a) (b)

Figure A.8 Yao Graph (a) and Undirected Yao Graph (b). In YG_6, directed edge (u, v) exists if and only if node v is the closest neighbor in one of the cones centered at u. Edges in YG_6 might be unidirectional (bold edges).

The RNG and GG of a set of points in the plane are reported in Figure A.7.

Theorem A.2.7 *Given a set N of points in the plane, we have*

$$EMST \subseteq RNG \subseteq GG \subseteq DT.$$

Definition A.2.8 (Yao graph) *Given a set N of points in the plane, and an integer $k \geq 6$, the Yao Graph of parameter k is the directed graph $YG_k = (N, E_k)$ defined as follows. At each node $u \in N$, divide the plane into k equally sized cones originating at u. Denoting by*

$C_u^1, \ldots, C_u^k$ the cones for node u, we have that $(u, v) \in E_k$ if and only if there exists cone C_u^i such that v is the closest neighbor of u in C_u^i.

Definition A.2.9 (Undirected Yao graph) *Given a set N of points in the plane, and an integer $k \geq 6$, the Undirected Yao Graph of parameter k is the graph $UYG_k = (N, E_k)$, where $(u, v) \in E_k$ if and only if either edge (u, v) or edge (v, u) is in YG_k.*

The YG and UYG of a set of points in the plane are reported in Figure A.8.