Solutions for Homework 4

3.50 (**Solution**): (a). Since $\{p_j\}$ is the stationary distribution, we have for all $j \in S$

$$p_j \left(\sum_{i \in \bar{S}} q_{ji} + \sum_{i \notin \bar{S}} q_{ji} \right) = \sum_{i \in \bar{S}} p_i q_{ij} + \sum_{i \notin \bar{S}} p_i q_{ij}$$

Using the given relation, we obtain for all $j \in \bar{S}$

$$p_j \sum_{i \in \bar{S}} q_{ji} = \sum_{i \in \bar{S}} p_i q_{ij}$$

Dividing by $\sum_{i \in \bar{S}} p_i$ on both sides, we have

$$\bar{p}_j \sum_{i \in \bar{S}} q_{ji} = \sum_{i \in \bar{S}} \bar{p}_i q_{ij}$$

for all $j \in \bar{S}$, showing that $\{\bar{p}_j\}$ is the stationary distribution of the truncated chain.

(b). If the original chain is time reversible, we have $p_j q_{ji} = p_i q_{ij}$ for all i and j, so the condition of part (a) holds. Therefore, we have $\bar{p}_j q_{ji} = \bar{p}_i q_{ij}$ for all states i and j of the truncated chain.

(c). The finite capacity system is a truncation of the two independent $M/M/1$ queues system, which is time-reversible. Therefore, by part (b0, the truncated chain is also time reversible. The formula for the steady state probabilities is a special case of eq. (3.39) of Section 3.4.

3.55 (**Solution**) Let $\rho_1 = \lambda_1/\mu_1$ and $\rho_2 = \lambda_2/\mu_2$. We can easily find the total arrival rates for CPU and I/O are $\lambda_1 = \lambda/p_1$ and $\lambda_2 = \lambda p_2/p_1$, respectively. Using Jackson's Theorem and equations (3.34)-(3.35), we obtain

$$p(n_1, n_2) = \begin{cases} p_0 \frac{(m\rho_1)^{n_1}}{n_1!} \cdot \rho_2^{n_2}(1-\rho_2), & n_1 \le m \\ p_0 \frac{m^m \rho_1^{n_1}}{m!} \rho_2^{n_2}(1-\rho_2), & n_1 > m \end{cases}$$

where

$$p_0 = \left[\sum_{n=0}^{m-1} \frac{(m\rho_1)^n}{n!} + \frac{(m\rho_1)^m}{m!(1-\rho_1)} \right]^{-1}$$

3.58 (**Solution**) We convert the system into a closed network with M customers as indicated in the hint. The $(K+1)$st queue corresponds to the "outside world". It is easy to see that the queues of the open systems are equivalent to the first k queues of the closed system. For example, when there is at least one customer in the $(k+1)$st queue (equivalently, there are less than M customers in the open system), the arrival rate at queue i is r_i, the routing probability from $(k+1)$st queue is $r_i/\sum_{i=1}^k r_i$, the service rate at the $(k+1)$st queue is $\sum_{i=1}^k r_i$. Furthermore, when the $(k+1)$st queue is empty no external arrivals can occur at any queue i, $i=1,2,\ldots,k$. If we denote with $p(n_1,\ldots,n_k)$ the steady state distribution for the open system, we get

$$p(n_1,n_2,\ldots,n_k)) = \begin{cases} 0 & \sum_{i=1}^k n_i > M \\ \frac{\rho_1^{n_1}\rho_2^{n_2}\cdots\rho_k^{n_k}\rho_{k+1}^{\left(M-\sum_{i=1}^k n_i\right)}}{G(M)} & \text{otherwise} \end{cases}$$

where

$$\begin{aligned} \rho_i &= \frac{r_i}{\mu},\ i=1,2,\ldots,k \\ \rho_{k+1} &= \frac{\sum_{i=1}^k r_i(1-\sum_{j=1}^k p_{ij})}{\sum_{i=1}^k r_i} \end{aligned}$$

and $G(M)$ is the normalizing factor.

3.61 (**Solution**) We have $\sum_{i=0}^m p_i = 1$. The arrival rate at the CPU is λ/p_0, and the arrival rate at the ith I/O port is $\lambda p_i/p_0$. By the Jackson's Theorem, we have

$$p(n_0,n_1,\ldots,n_m) = \prod_{i=0}^m \rho_i^{n_i}(1-\rho_i)$$

where for $i>0$. The equivalent tandem system will be the one with arrival rate λ, the service rate for queue 0 is $\mu_0 p_0$ and for the queue i ($i>0$) is $\mu_i p_0/p_i$.

3.64 (**Solution**) (a). The state is determined by the number of customers at node 1 (one could use node 2 just as easily). When there are customers at node 1 (which is the case for state 1,2 and 3), the departure rate from node 1 is μ_1, each such departure causes the state to decrease. When there are customers in node 2 (which is the case for states 0, 1 and 2), the departure from node 2 is μ_2, each departure causes the state to increase.

(b). Letting p_i be the steady state probability of state i, we have $p_i = p_{i-1}\rho$, where $\rho = \mu_2/\mu_1$. Thus, $p_i = \rho_0\rho^i$. Solving for p_0, we have

$$p_0 = [1+\rho+\rho^2+\rho^3]^{-1},\ p_i = p_0\rho^i,\ i=1,2,3.$$

(c). Customers leave node 1 at rate μ_1 for all states other than state 0. Thus, the time average rate at which customers leave node 1 is $\mu_1(1-p_0)$, which is

$$r = \frac{\rho+\rho^2+\rho^3}{1+\rho+\rho^2+\rho^3}\mu_1.$$

(d). Since there are three customers in the system, each customer cycles at one third the rate at which departures occur from node 1. Thus, a customer cycles at rate $r/3$.

(e). The Markov process is a birth-death process and thus reversible. What appears as a departure from node i in the forward process appears as an arrival to node i in the backward process. If we order the customers 1, 2 and 3 in the order in which they depart a node, and the note that this order never changes (because of the FCFS service at each node), then we see that in the backward process, the customers keep their identity, but the order is reversed with backward departures from node i in the order $3, 2, 1, 3, 2, 1, \ldots$.