Solutions for Homework 1

1. The exponential distribution for the random variable ξ has the following density function

$$f(t) = \lambda e^{-\lambda t},\ t \geq 0.$$

   (1). Find the mean, the variance and the coefficient of variation;

   (2). Find the characteristic function, moment generating function and the Laplace transform of the pdf;

   (3). Show that the exponential distribution has the following memoryless property: for any $s \geq 0$ and $t \geq 0$, we have

$$P(\xi \geq s+t | \xi \geq s) = P(\xi \geq t).$$

   In fact, a stronger version can be shown: for any nonnegative random variable ν, we have

$$P(\xi \geq t+\nu | \xi \geq \nu) = P(\xi \geq t).$$

   Solution: (1). The mean is $1/\lambda$, the variance is $1/\lambda^2$, and the coefficient of variation is 1.

   (2). The characteristic function is

$$\phi(t) = E(e^{jXt}) = \int_0^\infty e^{jxt}[\lambda e^{-\lambda x}]dx = \frac{\lambda}{-jt+\lambda}.$$

   The moment generating function is

$$M(v) = E[e^{vX}] = \frac{\lambda}{-v+\lambda}.$$

   The Laplace transform is

$$f^*(s) = E[e^{-sX}] = \frac{\lambda}{s+\lambda}.$$

   (3). $P(\xi > x) = e^{-\lambda x}$, we have

$$P(\xi \geq s+t | \xi \geq s) = \frac{P(\xi > t+s)}{P(\xi > s)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = P(\xi > t).$$

   The second statement has been proved in class.

2. Find the pdf for the minimum of K independent random variables, each of which is exponentially distributed with parameter λ. Find the Laplace transform of the pdf for the sum of the K independent random variables, each of which is exponentially distributed?

Solution: Let $X_1, X_2, \ldots, X_K$ denote the K independent exponential random variables with parameter λ, let X be the minimum of all, i.e., $X = \min\{X_1, X_2, \ldots, X_K\}$, let F denote the CDF, then we have

$$\begin{aligned} 1 - F(x) &= P(X > x) = P(X_1 > x, X_2 > x, \ldots, X_K > x) \\ &= P(X_1 > x)P(X_2 > x) \cdots P(X_K > x) = [P(X_1 > x)]^K = e^{-K\lambda x}. \end{aligned}$$

Thus, the pdf is

$$f(x) = k\lambda e^{-K\lambda x}.$$

which is exponentially distributed with parameter $K\lambda$. Let $Y = X_1 + \cdots + X_K$, then its Laplace transform will be

$$F_Y^*(s) = E(e^{-sY}) = E(e^{-sX_1}e^{-sX_2} \cdots e^{-sX_K})E[e^{-sX_1}]E[e^{-sX_2}] \cdots E[e^{sX_K}] = \left(\frac{\lambda}{s+\lambda}\right)^K.$$

3. Consider the discrete-time, discrete-state Markov chain with the probability transition matrix

$$\begin{pmatrix} 0.5 & 0.5 \\ 0.75 & 0.25 \end{pmatrix}.$$

Find the stationary state probability vector π. Suppose that you are an outsider observer and observe the evolution of the Markov chain, how often you find the chain stay in the state 1?

Solution: To find the stationary probability distribution, we only need to find the solution for the following equations

$$\begin{aligned} \pi P &= \pi \\ \pi e &= 1 \end{aligned}$$

where $\pi = (p_0, p_1)$ and $e = (1, 1)^T$. These equations can be explicitly written as

$$\begin{aligned} 0.5p_0 + 0.75p_1 &= p_0 \\ 0.5p_0 + 0.25p_1 &= p_1 \\ p_0 + p_1 &= 1 \end{aligned}$$

Solving these equations, we obtain $p_0 = 0.6$ and $p_1 = 0.4$. Thus, the stationary distribution is $\pi = (0.6, 0.4)$.

Thus, we will find the chain in state 1 40% of the time (the long-run probability).